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# On the transition from nonrenewable energy to renewable energy *

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**Abstract**

In this paper we use the CMM model (Chakravorty et al.,2006) in discrete time and obtain more results concerning the exhaustion time of Non-Renewable Resource (NRE), the dynamic regimes of energy prices, of the stocks of pollution. We show that NRE is exhausted in finite time and is directly influenced by the initial stock of NRE and the costs of NRE and RE. Higher is the initial stock of NRE, far is the time of exhaustion of NRE. Higher is the cost of NRE (resp. the difference of unit costs between RE and NRE), far is the time of exhaustion of NRE. Furthermore, we show that the abatement intervenes, when necessary, not more than two periods. We also show that, when the unit extraction cost of RE is not very high, the stocks of emissions will never be binding if and only if, the initial stock of NRE is less than a critical value.

**Keywords:** Dynamic optimization, Natural resources, Energetic transition, Environment.

**JEL Classification:** P28. Q01. Q32. Q42. Q48. Q52

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# 1 Introduction

This paper considers the combined effects of climate change mitigation, adaptation [1] and the scarcity of non-renewable energy (NRE) in presence of alternative renewable-energy (RE). A few existing literature has used combined approach (e.g. J. Byrne and C. Potanger, 2014). J. Byrne and C. Potanger, 2014 give conditions under which a given regime or policy can appear without confirming its apparition nor to explain things in detail (see for example Amigues et al. 2011, Chakravorty et al. 2006, Chakravorty et al. 2012 and Tahvonen and Salo 2001).
The mitigation policy is presented in our paper by the presence of a costly backstop-resource (i. e. RE), the possibility to abate with a constant average cost, and the natural dilution of environment that decreases the concentration of the Green-House-Gas (GHG) emissions. A a ceiling on the maximal stock of GHG is imposed exogenously. The energy-related adaptation is represented by the availability of NRE's stock at the beginning of the horizon, so we have to decide either to use it with a given consumption rate, or to storing it (to leave it in the ground).
For that, we consider the model in Chakravorty et al. 2006 but use discrete time. The economy represented by this model uses two energy resources, Non Renewable Ressource, NRE, and Renewable Ressource, RE, which are perfect substitutes and abatement $a_t$. The respective extraction (consumption) rates of NRE and RE are $x_t$ and $y_t$. The aggregate energy consumption is given by $q_t = x_t + y_t$, at time $t$. The felicity of the consumer is represented by a stationary utility function depending on the total energy consumption $q$. However, the consumer takes into account the energy extraction costs and the abatement cost.
We obtain more results than Chakravorty et al. 2006. First, we show that, for any initial stock of NRE, its exhaustion is in finite time. Concerning the exhaustion, the existing literature on the optimal allocation of NRE is divided into three points of view. Some authors claim that the final stock (non-economic reserve) of NRE depends on the initial stock and the level of industrialization (Tahvonen and Salo 2001 and others); other ones state that NRE should be used indefinitely (Hotelling 1931); and finally, those that claim that NRE will be exhausted in finite time but without a robust analytic proof (Chakravorty et al. 2006 and 2012).
Second, the exhaustion time of NRE's stock is directly influenced by the initial stock of NRE and the costs of NRE and RE. If the initial stock of NRE becomes very large, very far is the time of exhaustion of NRE. If the cost of NRE (resp.

1. The existing literature exhibits that effective climate change response requires both mitigation (reducing Green-House-Gas (GHG) production and sequestering carbon) and adaptation, e. g. preparing for future climate regimes. There is a spectrum of energy-related mitigation strategies including sequestering carbon, more efficient fossil-fuel combustion, improving energy efficiency, and transitioning to alternative energy source (e. g. solar, wind). Energy-related adaptation response includes upgrading network infrastructure to cope with higher demand of energy and/or damage from natural hazard (e. g. fire and floods), moving energy infrastructure away from vulnerable locations, storing oil to minimize disruption from supply failure, and developing smart-grids (J. Byrne and C. Potanger, 2014).

the difference of unit costs of using RE and NRE) becomes very large, very far is the time of exhaustion of NRE. More interestingly, we can refine these results in the case where the stock of pollution never reaches the exogenous ceiling. The exhaustion time increases either with the volume of the initial stock of NRE, or with the difference of the unit costs of using RE and NRE.
Third, it is not necessary to abate in most cases. And if abatement is required, we will not do that more than two periods.
Fourth, we completely describe the dynamic regimes of quantities of energy consumption, of energy prices and of the stocks of pollution.
Finally, we show that effectiveness of environmental policies depends on the level of the initial stock of NRE. When the unit extraction cost of RE is not very high, there is a critical value, $S_h$, of the initial stock of NRE under which no environmental policy will be effective because of the ceiling of the stocks of pollution will never be reached. And if the initial stock of NRE is above this critical value, there are some successive periods where the stock of pollution attains the ceiling and environmental policy will be required (i.e. abatement, simultaneous use of both energies).

The paper is organized as follows. In Section 2, we present the model and the optimality conditions. In Section 3, we present and discuss the properties of the optimal path. In Section 4 we present the possible dynamic regimes of quantities of energy consumption, of energy prices and of the stocks of pollution. In Section 5, we exhibit the influences of the level of NRE initial stock and of the extraction costs of NRE and RE on the exhaustion time of NRE. Concluding remarks are in Section 6. Section 7 is Appendix and contains all the proofs.

## 2 The model

The economy uses two energy resources, Non Renewable Ressource, NRE and Renewable Ressource, RE, which are perfect substitutes. Their respective extraction (consumption) rates are $x_t$ and $y_t$. The aggregate energy consumption is given by $q_t = x_t + y_t$, at time $t$. The instantaneous utility from energy consumption at time $t$ is $u(q_t)$.

**Assumption 1.** *$u : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{C}^1$, strictly increasing and strictly concave, i.e. $u'(q) > 0$, for all $q > 0$, $u'$ is strictly decreasing.*

**Assumption 2.** *$u$ satisfies Inada condition, i.e. $lim_{q\to 0}u'(q) = +\infty$.*

Initial reserves of NRE are assumed to be known and denoted by $S_{-1}$, the average extraction cost of NRE is supposed to be constant and denoted by $c_e$, and the pollution per unit of consumed NRE (coal for example) is given by $\zeta$. Let $S_t$ be the amount of the NRE energy available at time $t$, so that

$$S_{t-1} - S_t = x_t, \forall t \geq 0, S_{-1} > 0 \quad \text{given.} \tag{1}$$

Let $Z_t$ be the stock of pollution. The gross emission of pollution at time $t$, $z_t$, is assumed to be proportional to the consumption of NRE, $z_t = \zeta x_t$. The natural regeneration capacity of atmosphere is proportional to the stock of pollution,

$\alpha Z_t$, where $\alpha$ is some constant. Moreover, the stock of pollution can be reduced through costly abatement, whose instantaneous rate is given by $a_t$. Then the rate of change of $Z_t$ is given by

$$Z_t - Z_{t-1} = z_t - a_t - \alpha Z_t = \zeta x_t - a_t - \alpha Z_t, \forall t \geq 0, Z_{-1} > 0 \text{ given.} \tag{2}$$

The total abatement cost at time $t$ is given by $c_a a_t$, with $c_a$ the unit cost of abatement. The ceiling on pollution stock is $\bar{Z}$, and it is imposed by the regulator. Then

$$\bar{Z} - Z_t \geq 0, \forall t \geq 0. \tag{3}$$

The initial stock of pollution, $Z_{-1}$, is given and satisfies $Z_{-1} < \bar{Z}$. Define $c_r$ as the constant average extraction cost of RE.

### 2.1 The optimization problem

The objective of the economic planner is to maximize the inter-temporal utility

$$\max_{\{x_t, y_t, a_t\}} \sum_{t=0}^{+\infty} \beta^t [u(x_t + y_t) - c_e x_t - c_a a_t - c_r y_t] \tag{4}$$

under the constraints given by equations (1), (2) and (3), and the positivity constraints:

$$S_t \geq 0, x_t \geq 0, y_t \geq 0, a_t \geq 0, Z_t \geq 0. \tag{5}$$

The unit costs of extraction of (non renewable resource) NRE and (renewable resource) RE are respectively $c_e$, $c_r$, while the unit cost of abatement is $c_a$.

**Proposition 2.1.** *There exists an optimal solution.*

The Lagrangian of the problem is

$$\begin{aligned} L = \sum_{t=0}^{+\infty} \beta^t [u(x_t + y_t) - c_e x_t - c_a a_t - c_r y_t] &- \sum_{t=0}^{+\infty} \beta^t \lambda_t (x_t + S_t - S_{t-1}) \\ + \sum_{t=0}^{+\infty} \beta^t \mu_t (Z_t - Z_{t-1} - \zeta x_t + a_t + \alpha Z_t) &+ \sum_{t=0}^{+\infty} \beta^t \eta_t (\bar{Z} - Z_t) + \sum_{t=0}^{+\infty} \beta^t \gamma_t^x x_t \\ + \sum_{t=0}^{+\infty} \beta^t \gamma_t^a a_t + \sum_{t=0}^{+\infty} \beta^t \gamma_t^y y_t + \sum_{t=0}^{+\infty} \beta^t \gamma_t^s S_t &+ \sum_{t=0}^{+\infty} \beta^t \gamma_t^z Z_t \end{aligned} \tag{6}$$

The first-order conditions give

$$u^{'}(x_t + y_t) - c_e - \zeta \mu_t - \lambda_t + \gamma_t^x = 0 \tag{7}$$
$$u^{'}(x_t + y_t) - c_r + \gamma_t^y = 0 \tag{8}$$
$$-c_a + \mu_t + \gamma_t^a = 0 \tag{9}$$
$$\gamma_t^z + (1 + \alpha)\mu_t - \beta \mu_{t+1} - \eta_t = 0 \tag{10}$$
$$-\lambda_t + \beta \lambda_{t+1} + \gamma_t^s = 0 \tag{11}$$

where $\lambda_t, \mu_t, \eta_t, \gamma_t^x, \gamma_t^a, \gamma_t^y, \gamma_t^s, \gamma_t^z \geq 0$ are the Lagrangian multipliers and they are non negative.
The slackness conditions associated with this problem are:

$$\gamma_t^a a_t = 0; \gamma_t^a \geq 0; \tag{12}$$
$$\eta_t(\bar{Z} - Z_t) = 0; \eta_t \geq 0; \tag{13}$$
$$\gamma_t^x x_t = 0; \gamma_t^x \geq 0; \tag{14}$$
$$\gamma_t^s S_t = 0; \gamma_t^s \geq 0; \tag{15}$$
$$\gamma_t^y y_t = 0; \gamma_t^y \geq 0; \tag{16}$$
$$\gamma_t^z Z_t = 0; \gamma_t^z \geq 0; \tag{17}$$
$$\lambda_t(x_t + S_t - S_{t-1}) = 0; \tag{18}$$
$$\mu_t(Z_t - Z_{t-1} - \zeta x_t + a_t + \alpha Z_t) = 0 \tag{19}$$

We add the following assumption

**Assumption 3.** $c_e < c_r$

Assumption 3 is very natural. If the optimal solutions $(x_t, y_t)$ are both positive, the FOC (7), (8) give $c_e \leq u'(x_t + y_t) \leq c_r$.

# 3 Properties of the optimal path

## 3.1 The extraction path of NRE

In this part we describe the characteristics of the dynamics extraction of NRE. The following proposition shows that, in the optimal extraction of NRE, we cannot alternate between the use and the non-use of NRE.

**Proposition 3.1.** *We cannot alternate the use and non-use of NRE. In other words, if there exists $t_1 \geq 0$ such that $x_{t_1} = 0$, then $x_t = 0$ $\forall t \geq t_1$.*

Thus it will be impossible to switch to the only use of RE and then go back to NRE. This means that once the energy transition from non-renewable-energy to renewable-energy is achieved, we will use the only renewable-energy for ever. The following proposition confirms the existence of this switch date $T$.

**Proposition 3.2.** *There exists an instant $t_s \geq 0$ such that $x_{t_s} = 0$.*

We denote by $T$ the switch date, i. e. the first instant $T \geq 0$ such that $x_T = 0$. We denote by $\bar{S}$ the non-economical reserve of NRE, i.e. the stock of NRE that will never be extracted.

**Proposition 3.3.** *After the switch date, $T$*

$$\mu_t = 0, a_t = 0, \forall t \geq T. \tag{20}$$

**Remark 3.1.** *This result is intuitive because once we stop using NRE we will no longer pollute, it is not necessary to abate.*

**Lemma 3.1.** *The shadow price $\lambda_t$ is strictly positive for any $t \geq 0$.*

The FOC (11) gives

$$\lambda_t = \beta\lambda_{t+1} + \gamma_t^s$$

If the stock of NRE is never exhausted ($S_t > 0$ for any $t$) then we obtain the traditional Hotelling rule ("the shadow price of energy must grows at the rate of the interest")

$$\lambda_t = \beta\lambda_{t+1}.$$

What differs in this model is that there are other shadow prices that also drive the energy price.
The first main result of this paper is the following theorem:

**Theorem 1.** *There is a date $T$ such that $x_t > 0$ for any $t < T$ and $x_t = 0$ for $t \geq T$. Moreover, the stock of the NRE will be exhausted ($\bar{S} = 0$) at date $T-1$.*

**Remark 3.2.** *This result is new with regard to the existing literature, since it confirms the exhaustion, in finite time, of NRE whatever the initial conditions.*

Despite the importance of this result, $\bar{S} = 0$, it can be considered as intuitive because of the marginal extraction cost of NRE is lower than that of RE ($c_e < c_r$) and the presence of the natural dilution of environment.

The importance of necessity to switch to the clean and sustainable energies does not mean that one has to suddenly stop using NER by keeping it in the ground, but it means that one must switch to the only use of RE after exhaustion of NRE by following a given optimal use of both energies. The properties of this optimal sequence are described in what follows.

**Proposition 3.4.** *One can abate only if the stock of pollution is at its maximum, i. e. if $a_t > 0$ then $Z_t = \bar{Z}$.*

The following proposition shows an important and new result. It concerns the conditions under which we can abate (Proposition 3.3 gives that there is no abatement for $t \geq T$).

**Proposition 3.5.** *If $a_t > 0, x_t > 0$ for some $t$, then $c_r > c_e + \zeta c_a$.*

$\zeta c_a$ is the cost of abatement or the cost of pollution when the stock of pollution is at its maximum. We can abate only when the unit cost of RE is larger than the total cost of consuming NRE (cost of extraction, $c_e$, + cost of pollution, $\zeta c_a$, generated by this consumption of NRE).This result is interesting, and means that the abatement can occur only if the renewable-energy is relatively expensive compared with both costs of extraction of NRE and the pollution. However, the condition $c_r \geq c_e + \zeta c_a$ is not sufficient to have $a_t > 0, x_t > 0$ as it is shown in the following proposition.

**Proposition 3.6.** *Assume $c_e + \zeta c_a > u'(\frac{\alpha\bar{Z}}{\zeta})$. Then $a_t = 0$ for all $t$.*

The interpretation of the previous result is as follows. Let us recall that $\frac{\alpha\bar{Z}}{\zeta}$ is the maximum of consumption of NRE when the pollution reaches the ceiling. This consumption is obtained when, in particular, we have no abatement.

Hence, $u'(\frac{\alpha\bar{Z}}{\zeta})$ is the largest marginal satisfaction of the consumer when the pollution is maximal. When the total cost of consuming NRE (cost of extraction, $c_e$, + cost of pollution, $\zeta c_a$, generated by this consumption of NRE) is larger than the maximal marginal satisfaction of consuming NRE when the pollution reaches the ceiling, there will be no abatement. Indeed, abatement will increase the consumption and lower the marginal satisfaction.

Equation (9) can be rewritten as

$$c_a = \mu_t + \gamma_t^a$$

That means that the average cost of abatement must be equal to the aggregate abatement profit. The aggregate abatement profit is the sum of marginal benefit of abatement and the profit from renouncing to abate.

**Proposition 3.7.** *It is not optimal to have a totally clean environment at some date. At the optimum, one has $Z_t > 0, \forall t$.*

**Remark 3.3.** *This new result means that the objective of the conservation of the climate is not to have a completely clean environment. To have a completely clean climate is even against the well-being of the society.*

In what follows we examine the conditions under which we can have the simultaneous use of both energies. We know that for $t \geq T$ the only RE is used for ever. Thus in the following we consider only the periods $t$ such that $0 \leq t < T$. More precisely we consider $0 \leq t \leq T-2$ because, at the instant of switch to the only use of RE, we cannot confirm the following propositions.

**Proposition 3.8.** *At any date $t \leq T-2$, we can have the simultaneous use of both energies only if the stock of pollution is at its maximum, i. e, $y_{T-t} > 0 \implies Z_{T-t} = \bar{Z}$ for $t \geq 2$.*

The simultaneously use of costly RE and cheap NRE can only be realized to respond to extreme environmental problems. Once the regime of simultaneous use of both energies occurs, it will hold until exhaustion of NRE. Using RE as a temporary solution of environmental problems is not optimal.

**Proposition 3.9.** *Once we use RE, we will use it forever. If $y_\tau > 0, \tau < T$, then $y_t > 0$ $\forall t \geq \tau$.*

Despite the fact that RE can respond to environmental problems, the simultaneous use is not always possible. The following proposition shows that.

**Proposition 3.10.** *Assume $c_r \geq u'(\frac{\alpha}{\zeta}\bar{Z})$. Then $x_t > 0,\ y_t = 0$ for $t \leq T-2$.*

**Remark 3.4.** *This result was also found by CMM.*

Denote $\bar{c} = u'(\frac{\alpha}{\zeta}\bar{Z})$. It is clear that if the environmental problems are important, i. e, $\bar{Z}$ is small, $\alpha$ is small or $\zeta$ is high, then $\bar{c}$ is high. And if the environmental damages are not very important then $\bar{c}$ is smaller. In the first case the chance to have simultaneous use is high, and it is low in the second case. Thus to have simultaneous use we have to impose a low $\bar{Z}$, since we cannot change the rate of natural dilution of environment nor to change easily the marginal unit of pollution due to the use of NRE.

The following proposition shows that we cannot fight environmental problems with both abatement and the use of RE. And from proposition 3.8, the use of RE is always the latest choice to fight environmental problems.

**Proposition 3.11.** *Assume $c_r > c_e + \zeta c_a$, and let $\tau$ be the first instant where $y_\tau > 0$, $\tau \leq T-3$, if it exists. Then $a_t = 0$, for $\tau + 1 \leq t \leq T-2$.*

**Corollary 3.1.** *When abatement occurs, then RE is not used, i.e. if $a_t > 0, t \leq T-2$ then $y_t = 0, \forall t \leq T-3$.*

**Proposition 3.12.** *Once we pass a phase of ceiling binding, we will no more have a second ceiling binding phase and the NRE extraction will be lower than $\alpha\bar{Z}/\zeta$. In other terms, if $x_t > 0, Z_{t-1} = \bar{Z}, Z_t < \bar{Z}$, then $x_\tau < \alpha\bar{Z}/\zeta$, $Z_\tau < \bar{Z}, \forall \tau \geq t$*

**Proposition 3.13.** *Before the ceiling binding period, the only NRE is used without ceiling binding and with a strictly decreasing extraction, but all are greater or equal to $\alpha\bar{Z}/\zeta$. In other terms, if $x_t > 0, Z_{t-1} < \bar{Z}, Z_t = \bar{Z}$, then $x_t \geq \alpha\bar{Z}/\zeta$, $x_\tau > x_{\tau+1}$, and $Z_\tau < \bar{Z}$, $\forall \tau = 0, \ldots, t-1$*

The following proposition shows that, in the case when $c_r \leq u'(\frac{\alpha}{\zeta}\bar{Z})$, if the stock of pollution reaches its upper level at a given instant $t$ $(0 \leq t \leq T)$, it will still binding until the exhaustion of NRE stock.

**Proposition 3.14.** *Assume $c_r \leq u'(\frac{\alpha}{\zeta}\bar{Z})$.*
*If $Z_t = \bar{Z}$, $t \leq T-2$, then $Z_\tau = \bar{Z}$ for $t \leq \tau \leq T-1$.*

The following proposition shows that, under the condition $c_r \geq c_e + \zeta c_a$, if there is a phase or regime with abatement, it will occur before simultaneous use of both energies or before the only use of NRE with ceiling binding. It must be used together with propositions 3.8 and 3.9. Furthermore, it shows that we cannot abate more than two period.

**Proposition 3.15.** *Assume $c_r \geq c_e + \zeta c_a$. If we have two consecutive periods of ceiling binding, and if there is abatement at the second, then the ceiling was not binding before these two periods. In other terms, if we have for $0 \leq t \leq T-2$ that $Z_t = \bar{Z}$ and $Z_{t+1} = \bar{Z}, a_{t+1} > 0$, then $Z_{t-1} < \bar{Z}$.*

Combining Proposition 3.14 and Proposition 3.4, we get the following corollary.

**Corollary 3.2.** *Assume $c_r \geq c_e + \zeta c_a$. The abatement, if it occurs, cannot last more than two periods.*

We can say that introducing abatement to fight environmental problems is of very little importance.

## 4 Succession of regimes

In this section we exhibit the possible regimes of extraction of both energies, of the enegy prices and of the stocks of pollution.

**Theorem 2.** *We have*
*(i) either $Z_t < \bar{Z}$, for all $t$,*
*(ii) or there exists $t < T$ such that $Z_t = \bar{Z}$. Let $\tau$ is the first date when $Z_t = \bar{Z}$. In this case we have three regimes.*

- $Z_t < \bar{Z}$, $\forall t \leq \tau - 1$
- $Z_t = \bar{Z}, \forall t \in \{\tau, \dots, \tau'\}$, $\tau \leq \tau' \leq T-1$. *If* $c_r \leq u'(\frac{\alpha}{\zeta}\bar{Z})$, *then* $\tau' = T-1$.
- $Z_t < \bar{Z}, \forall t > \tau'$

One may wonder whether the regime with $Z_t < \bar{Z}, \forall t$, exists. The answer is yes, when $S_{-1}$ is low enough. More precisely,

**Proposition 4.1.** *If $S_{-1} < \frac{\alpha\bar{Z}}{\zeta}$ then $Z_t < \bar{Z}$ for all $t$.*

We now show that, under an additional condition, for any $S_{-1}$ high enough, there exists a date $\tau < T$ [2] such that $Z_\tau = \bar{Z}$.

**Proposition 4.2.** *Assume $c_r < u'(\frac{\alpha\bar{Z}}{\zeta})$. Let*

$$\bar{T} = \frac{u'^{-1}(c_r) + \bar{Z}/\zeta}{u'^{-1}(c_r) - (\bar{Z}\alpha)/\zeta}, \ \bar{S} = \frac{\bar{Z}}{\zeta}(\alpha\bar{T} + 1)$$

*then*

$$S_{-1} > \bar{S} \Rightarrow Z_t = \bar{Z}, \text{ for some } t$$

In theorem 3, we show there exists a critical value for the initial stock of NRE when $c_r < u'(\frac{\alpha\bar{Z}}{\zeta})$. If the initial stock is lower than this critical value then the stock of pollution will never attain the ceiling. If the initial stock is higher than this critical value, then the stock of pollution will reach the ceiling at some date.

**Theorem 3.** *Assume $c_r < u'(\frac{\alpha\bar{Z}}{\zeta})$. There exists a critical value of the initial stock of non renewable energies, $S^h_{-1} < +\infty$, such that for each initial stock lower than this value, the upper bound of pollution stock will never be attained and if the initial stock of is higher than this value than there exists at least one period $t$ with $Z_t = \bar{Z}$. In other terms*

$$S_{-1} < S^h_{-1} \implies Z_t < \bar{Z}, \forall t \geq 0$$
$$S_{-1} > S^h_{-1} \implies Z_t = \bar{Z}, \text{ for at least one } t.$$

We completely characterize the optimal paths in Theorems 4 and 5.

2. $\tau < T$ since for any $t \geq T, Z_t < \bar{Z}$

**Theorem 4.** *Assume $Z_t < \bar{Z}, \forall t$. Then*

$$
\begin{aligned}
&(a)\ a_t = 0,\ \forall t \geq 0\\
&(b)\ x_t > 0, \forall t \leq T-1,\ x_t = 0,\ \forall t \geq T\\
&(c)\ Z_{T+t} = \frac{Z_{T-1}}{(1+\alpha)^{t+1}},\ \forall t \geq 0\\
&(d)\ y_t = 0,\ \forall t \leq T-2,\ u'(y_t) = c_r,\ \forall t \geq T\\
&(e)\ Z_t(1+\alpha) - Z_{t-1} = \zeta x_t,\ \forall t \leq T\\
&(f)\ x_0 > x_1 > \ldots > x_{T-1} > x_T = 0\\
&\quad u'(x_0) < u'(x_1) < \ldots < u'(x_{T-2}) < u'(x_{T-1} + y_{T-1}) \leq c_r = u'(y_s),\ \forall s \geq T.
\end{aligned}
$$

**Comments**

When $Z_t < \bar{Z}$ for any $t$, abatement is not required, we only use NRE energy until the date $T-2$. Both energies may be used but only at period $T-1$. The total energy consumptions decrease strictly from $t=0$ to $t=T$ and become constant after. The energy prices increase strictly from $t=0$ to $t=T$. It equals the unit extraction cots of RE for $t \geq T$. The stocks of pollution decrease strictly after $T$ and converge to zero when $T$ goes to infinity.

**Theorem 5.** *Assume $Z_t = \bar{Z}$, for some $t$. Let $\tau < T$ be the first date when $Z_t = \bar{Z}$ and $\tau' < T$ the biggest date with $Z_t = \bar{Z}$. Then*

$$
\begin{aligned}
&(a)\ Z_t = \bar{Z},\ \forall t \in \{\tau, \ldots, \tau'\}, \tau \leq \tau' \leq T-1\\
&(b)\ y_t = 0,\ \forall t < \tau\\
&(c)\ x_0 > x_1 > \ldots > x_{\tau-1} > x_\tau + y_\tau
\end{aligned}
$$

*I. Add the assumption $c_r < u'(\frac{\alpha\bar{Z}}{\zeta})$. Then*

$$
\begin{aligned}
&(a')\ \tau' = T-1\\
&(d)\ y_t > 0,\ \forall t \in \{\tau+1, \ldots, T-1\}\\
&(e)\ u'(x_0) < u'(x_1) < \ldots < u'(x_{\tau-1}) < u'(x_\tau + y_\tau) \leq c_r\\
&= u'(x_{\tau+1} + y_{\tau+1}) = \ldots = u'(x_{T-1} + y_{T-1}) = u'(y_s),\ \forall s \geq T
\end{aligned}
$$

*Add the assumption* $c_r \leq c_e + \zeta c_a$. *Then*

$(f)\ a_t = 0,\ \forall t$

*Replace the assumption* $c_r \leq c_e + \zeta c_a$ *by* $c_e + \zeta c_a < c_r$. *Then*

$(f')\ a_t = 0,\ \forall t \in \{\tau+2, \ldots, T-1\}$

*II. Replace Assumption* $c_r < u'(\frac{\alpha\bar{Z}}{\zeta})$ *by* $c_r \geq u'(\frac{\alpha\bar{Z}}{\zeta})$*. Then*

$(d')\ y_t = 0,\ \forall t \leq T-2$

*Add the assumption* $c_r \leq c_e + \zeta c_a$*. Then*

$(f)\ a_t = 0,\ \forall t$

$$(e')\ u'(x_0) < \ldots < u'(x_\tau) < u'(x_{\tau+1}) = \ldots = u'(x_{\tau'}) = u'(\frac{\alpha\bar{Z}}{\zeta})$$
$$< u'(x_{\tau'+1}) < \ldots < u'(x_{T-2}) < u'(x_{T-1} + y_{T-1}) \leq c_r = u'(y_s),\ \forall s \geq T$$

*Replace the assumption* $c_r \leq c_e + \zeta c_a$ *by* $c_e + \zeta c_a < c_r$*. Then*

$$(e'')\ \ u'(x_0) < \ldots < u'(x_\tau) < u'(x_{\tau+1}) \leq u'(x_{\tau+2}) = \ldots = u'(x_{\tau'})$$
$$< u'(x_{\tau'+1}) < \ldots < u'(x_{T-2}) < u'(x_{T-1} + y_{T-1}) \leq c_r = u'(y_s),\ \forall s \geq T$$

**Comments** When $Z_t = \bar{Z}$ for some $t$:
(i) We will not use RE before the first date when the stock of pollution reaches the ceiling.
(ii) If the unit extraction cost of RE is not high, namely less than $u'(\frac{\alpha\bar{Z}}{\zeta})$, then the ceiling-binding phase will end at period $T-1$. Both energies will be used between the second period of the ceiling-binding phase and the end of this phase. The energy prices increase strictly from $t = 0$ to the first date of the ceiling-binding phase. It equals the unit cost of RE extraction after beginning of this phase, i.e., even before the NRE exhaustion date. If moreover the RE extraction cost is less than the total NRE unit cost (NRE extraction cost+ pollution cost due to the use of this energy), then no abatement will be required from the third period of the ceiling-binding phase to $T-1$ (the date of the end of this phase). Abatement may be required for the two first periods of the ceiling-binding phase.
(iii) When the RE extraction cost is higher than $u'(\frac{\alpha\bar{Z}}{\zeta})$ but less than the total NRE cost then no abatement will be required. The prices will increase from $t = 0$ to the second period of the ceiling-binding phase, becomes constant until the end of the ceiling-binding phase, increases again after this date until the NRE exhaustion date and remains indefinitely equal to RE extraction cost after the NRE exhaustion date. However, if the RE extraction cost is higher than the total NRE cost, then the prices increase from $t = 0$ to the second period of the ceiling-binding phase, becomes constant from this second period to the end of the ceiling-bing phase and as just before, increases again after this end date until the NRE exhaustion date and remains indefinitely equal to RE extraction cost after the NRE exhaustion date. Abatement may be required for the two first periods of the ceiling-binding phase.

# 5 About the exhaustion date $T$

In this Section we would like to examinee the influences of the initial stock of NRE $S_{-1}$ and of the unit costs $c_r$, $c_e$ on the exhaustion date $T$. We neglect the role of $c_a$ since we have shown that the abatement intervenes at most two consecutive periods. Its role is not very important in the transition from NRE to RE.

**Theorem 6.** *We have*
*(i)* $S_{-1} \to +\infty \Rightarrow T \to +\infty$.
*(ii) Assume* $c_e \to +\infty, c_r \to +\infty$ *and still satisfy* $c_e < c_r$. *Then* $T \to +\infty$.
*(iii) We have*

$$c_r - c_e \to +\infty \implies T \to +\infty$$

**Remark 5.1.** *In (ii) of the previous theorem, since* $c_e$ *is very high, when we use NRE, the extraction at each period is very small. Since the total resource* $S_{-1}$ *is fixed, the exhaustion time* $T$ *becomes also very large. Assertion (iii) states if the difference of unit costs between RE and NRE becomes very high then the exhaustion time* $T$ *is also very high.*

In the case $Z_t < \bar{Z}$ for any $t$, e.g. $c_r < u'(\frac{\alpha\bar{Z}}{\zeta})$ and $S_{-1} < S^h_{-1}$ ($S^h_{-1}$ is the critical value for $S$, theorem 3), we can refine our results.

**Theorem 7.** *Assume* $Z_t < \bar{Z}$ *for any* $t$. *Let* $T'$ *be the exhaustion time associated with the stock* $S'_{-1}$. *Then*

$$S'_{-1} > S_{-1} \Rightarrow T' \geq T$$

**Theorem 8.** *Assume* $Z_t < \bar{Z}$ *for any* $t$.
*Assume* $(c'_r, c'_e)$ *satisfy* $c'_r - c'_e > c_r - c_e$, *and the sequence* $Z'_t$ *associated with* $(c'_r,\ c'_e)$ *still verifies* $Z'_t < \bar{Z}$ *for any* $t$. *Let* $T'$ *be the exhaustion time associated with* $(c'_r, c'_e)$. *We have* $T' \geq T$.

# 6 Concluding remarks

In this paper we deepen the CMM model of Chakravorty et al., 2006, by using discrete time and obtain more results. Backstop energy, a ceiling on the stock of GHG emission and the possibility to abate are introduced to the standard Hotelling model.
We show that non-renewable energy's stock will be used until its exhaustion in finite finite. Once we use RE, we will use it for ever. And when simultaneous use of NRE and RE take place, then the stock of pollution is at its maximum which was not reached before. However, the simultaneous use of both energies is not always possible. If the unit cost of RE is higher than maximal marginal satisfaction of consuming NRE when pollution reaches the ceiling, then simultaneous use of both energies cannot occur. Furthermore, when RE is used, abatement cannot occur. Thus, it is not optimal to jointly use these two policies (RE and abatement) to fight environmental problems. Abatement will not occur in most cases. Notably, abatement cannot occur when the total cost of NRE (cost of extraction of NRE+ the cost of pollution) is lower than the cost of RE. It cannot occur also if the total cost of NRE is larger than the maximal consumption satisfaction of consuming NRE when the stock of pollution is at its maximum. Adjusting GHG emissions to fight environmental problems is important only if the stock of NRE is sufficiently high, i.e. there exist a critical value of the initial stock such that if the initial stock is strictly lower than this value, there is no

a serious environmental damages which require environmental regulations. In this case, we have a pure Hotelling model, with only use of NRE with a strictly decreasing consumption's rate of NRE until its exhaustion. The net-energy's price is strictly increasing at a rate equal to the interest rate.
If the stock of NRE is larger than the critical value, then the stock of pollution reaches its upper bound at some periods. Before the ceiling-binding date, only NRE is used with strictly increasing energy's price. During the ceiling-binding phase, if the unit cost of RE is less than some value, say $\tilde{c}_r$, we will have simultaneous use of NRE and RE until exhaustion of NRE's stock. The energy's price is constant and equal to the RE's cost.
If the unit cost of RE is higher than $\tilde{c}_r$, then only NRE is used during the ceiling-binding phase. We can eventually require abatement but not more than two periods. The abatement, when it is required, will intervene at the beginning of the ceiling-binding phase. After that, energy's price will be constant until the end of ceiling-binding phase. The end of this phase can arrive before the exhaustion of NRE. After the ceiling-binding phase, we will have a pure Hotelling phase until exhaustion of NRE.
Regarding the exhaustion time of NRE, it always goes to infinity when the stock of NRE goes to infinity. In the case where the stocks of pollution will never bind, the exhaustion time increases with the size of the NRE stock.
The marginal costs of extraction of NRE and RE have also direct effects on the exhaustion time. If the costs of extraction of NRE, or of RE are very high then the exhaustion time becomes also very high. If the difference of unit costs between RE and NRE becomes very high, then the exhaustion time is very high.

# References


[1] Amigues J., Ayong Le Kama A., Chakrovarty U., and Moreaux M., Equilibrium Transitions from Non Renewable Energy to Renewable Energy under Capacity Constraints, *Working Paper*, (January 2011).

[2] Chakrovarty U., Leach A. and Moreaux M., Cycles in nonrenewable resource prices with pollution and learning-by-doing, *Journal of Economic Dynamics & Control*, 36 (2012) 1448-1461.

[3] Chakrovarty U., Magné B. and Moreaux M., A Hotelling model with a ceiling on the stock of pollution, *Journal of Economic Dynamics & Control*, 30 (2006) 2875-2904.

[4] Dasgupta P. and Heal G., The optimal deplation of exhaustible resources, *Energy Economics*, 23(2001) 619-636.

[5] Heal G., Reflections - The economics of renewable energy in the United States, *Review of Environmental Economics and Policy*, volume 4, issue 1, (Winter 2010), pp. 139-154.

[6] Hotelling H., The economics of exhaustible resources,*The Journal of Political Economy*, 39, (April 1931).

[7] Schumacher I., When should we stop extracting nonrenewable resources?, *Macroeconomic Dynamics.*,Vol. 15 Issue 4, (Sep 2011), pp. 495-512.

[8] Solow R., The Economic of Resources or the Resources of Economics, *American Economic Association*, (May 1974).

[9] Tahvonen O. and Salo S., Economic growth and transition between renewable and non-renewable energy resources *European Economic Review*, 45 (2001) 1379-1398.

# 7 Appendix

## 7.1 Proof of proposition 2.1

*Proof.* Let $(x_t, y_t, a_t, S_t, Z_t)_{t\geq 0}$ be optimal. Then at any $t$, $y_t$ must satisfy

$$u(x_t + y_t) - c_r y_t \geq u(x_t + y) - c_r y, \ \forall y \geq 0$$

Let $\varphi_t(y) = u(x_t + y) - c_r y$. The function $\varphi_t$ is strictly concave and satisfies $\varphi_t(0) = u(x_t) \geq 0$, $\varphi_t(+\infty) = -\infty$.
If $u'(x_t) > c_r$ then $y_t > 0$ and satisfies $u'(x_t + y_t) = c_r$. This is equivalent to $y_t = u'^{-1}(c_r) - x_t \leq u'^{-1}(c_r)$.
If $u'(x_t) \leq c_r$ then $y_t = 0$. In both cases, $y_t \in [0, u'^{-1}(c_r)]$. We can restrict $y_t$ in the interval $[0, u'^{-1}(c_r)]$. For the other variables we have

$$\begin{aligned} &0 \leq x_t \leq S_{-1} \\ &0 \leq S_t \leq S_{-1} \\ &0 \leq a_t \leq \zeta S_{-1} + \bar{Z} \\ &0 \leq Z_t \leq \bar{Z} \end{aligned}$$

The variables are in compact set for the product topology. Since the constraints are continuous functions of these variables, the feasible set is a compact set for the product topology. On the set of feasible sequences we have, for any $t$,

$$-c_e S_{-1} - c_a(\zeta S_{-1} + \bar{Z}) - c_r u'^{-1} \leq u(x_t + y_t) - c_e x_t - c_a a_t - c_r y_t \leq u(S_{-1} + u'^{-1})$$

Since $\beta \in (0, 1)$, the inter temporal utility is continuous for the product topology. Existence of optimal paths follows the Weirstrass Theorem. □

## 7.2 Proof of proposition 3.1

*Proof.* Assume $x_t = 0$ and $x_{t+1} > 0$, then $S_t = S_{t-1}$ (because we have $x_t = S_{t-1} - S_t$.

We have $S_t > 0$ (if not $x_{t+1} = S_t - S_{t+1} = -S_{t+1} \leq 0$) and $u^{'}(y_t) = c_r$ (since $y_t > 0$, from Assumption 2), that gives from (7)and (8))

$$\begin{aligned} u^{'}(y_t) &= c_e + \zeta\mu_t + \lambda_t - \gamma_t^x \quad \text{and} \\ u^{'}(x_{t+1} + y_{t+1}) &= c_r - \gamma_t^y = c_e + \zeta\mu_{t+1} + \lambda_{t+1}. \end{aligned}$$

We have from equation (2) $Z_t - Z_{t-1} - \zeta x_t + a_t + \alpha Z_t = 0$. That implies $(1+\alpha)Z_t + a_t = Z_{t-1} + \zeta x_t = Z_{t-1}$. Thus, $Z_t \leq \frac{Z_{t-1}}{1+\alpha} \leq \frac{\bar{Z}}{1+\alpha} < \bar{Z}$, and hence,

$\eta_t = 0$.
Equation (10) gives $\beta\mu_{t+1} = (1+\alpha)\mu_t + \gamma_t^z \geq (1+\alpha)\mu_t$. That implies

$$\mu_{t+1} \geq \frac{1+\alpha}{\beta}\mu_t. \tag{21}$$

And equations (11) and (15) give $S_t > 0$. That gives $\gamma_t^s = 0$. Thus

$$\lambda_{t+1} = \frac{\lambda_t}{\beta} \geq \lambda_t. \tag{22}$$

On other hand, we cannot have $\lambda_t = 0, \mu_t = 0$, because otherwise we will have $c_r = u^{'}(y_t) = c_e - \gamma_t^x < c_r$ which is impossible. Thus we have either $\lambda_t > 0$ or $\mu_t > 0$. That implies that we have either $\lambda_{t+1} > \lambda_t$ or $\mu_{t+1} > \mu_t$ (from equations (21) and (22)).
From what precedes we have

$$\begin{array}{rcl} c_r \geq c_r - \gamma_{t+1}^y & = & u^{'}(x_{t+1} + y_{t+1}) \\ & = & c_e + \zeta\mu_{t+1} + \lambda_{t+1} \\ & > & c_e + \zeta\mu_t + \lambda_t - \gamma_t^x \\ & = & u^{'}(y_t) = c_r, \end{array}$$

yielding a contradiction.
Hence $x_{t+1} = 0$. By induction $x_\tau = 0$, for any $\tau > t$. □

### 7.3 Proof of proposition 3.2

*Proof.* Suppose $x_t > 0, \forall t \geq 0$. Then we have $S_t > 0, \gamma_t^s = 0, \forall t \geq 0$.
Since $0 < S_{t+1} \leq S_t$ we have $S_t \to \bar{S} \geq 0, x_t \to 0$, as $t \to +\infty$ (because $x_t = S_{t-1} - S_t$). Observe that

$$\begin{array}{rcll} Z_t(1+\alpha) & \leq & a_t + Z_t(1+\alpha) & \\ & = & Z_{t-1} + \zeta(S_{t-1} - S_t) & \text{from equation (2)} \\ & \leq & \bar{Z} + \zeta(S_{t-1} - S_t) & \end{array}$$

Choose $\epsilon > 0$, small enough such that $u^{'}(\epsilon) > c_r$ and $\frac{\bar{Z}}{1+\alpha} + \frac{\zeta\epsilon}{1+\alpha} < \bar{Z}$. For $t$ large enough, say $t \geq T$, we have

$$\begin{array}{rcl} Z_t & \leq & \dfrac{\bar{Z}}{1+\alpha} + \dfrac{\zeta\epsilon}{1+\alpha} < \bar{Z} \\ x_t & < & \epsilon \\ \eta_t & = & 0 \quad \text{from equation (13)} \end{array}$$

and hence

$$\begin{array}{rcl} \mu_{t+1} & \geq & \dfrac{1+\alpha}{\beta}\mu_t \quad \text{from equation (10)} \\ \lambda_{t+1} & = & \dfrac{\lambda_t}{\beta} \quad \text{from equation (11)} \end{array}$$

Hence, for any $t \geq t, u^{'}(x_t) > u^{'}(\epsilon) > c_r$, that gives $y_t > 0, u^{'}(x_t + y_t) = c_r$ (because of $u^{'}(x_t + y_t) \leq c_r$ from equation (8)). From $u^{'}(x_t + y_t) = c_e + \zeta\mu_t + \lambda_t$ (equation (7)), we cannot have $\mu_t = \lambda_t = 0$ because $c_e < c_r$. Thus for $t \geq T$, we have either $\mu_t > 0$, or $\lambda_t > 0$.
If $\lambda_t > 0, \forall t \geq T$, then $\lambda_t \to +\infty$ as $t \to +\infty$. And we have a contradiction:

$$c_r = u^{'}(x_t + y_t) > \lambda_t \to +\infty.$$

And if $\mu_t > 0, \forall t \geq T$, this implies $\mu_t \to +\infty$ as $t \to +\infty$.We then obtain also a contradiction:

$$c_a > \mu_t \to +\infty$$

Hence, we conclude that there exists $t_s$ with $x_{t_s} = 0$. From proposition 3.1, $x_t = 0, \forall t \geq t_s$. □

### 7.4 Proof of proposition 3.3

*Proof.* Suppose $x_{t^s} = 0$, then $Z_t < \bar{Z}, \forall t \geq t^s$ (from propositions 3.1 and 3.2, and equation (2)). That implies $\eta_t = 0, \forall t \geq t^s$ (from equation (13)). Hence

$$\mu_{t+1} \geq \frac{(1+\alpha)}{\beta}\mu_t, \forall t \geq t^s \quad \text{(from equation (9))}.$$

Thus, if $\mu_{t^s} > 0$ for some $t \geq t^s$, then $\mu_t \to +\infty$ (from equations (9) and (12)), that is impossible, since $\mu_t \leq c_a$. Hence $\mu_t = 0, \forall t \geq t^s$.
On other hand, if $a_t > 0$, for some $t \geq t^s$ then $\gamma_t^a = 0$ (from equation (12)). That gives $\mu_t = c_a > 0$, that is impossible because $\mu_t = 0$ for any $t$. Hence $a_t = 0, \forall t \geq t^s$.

□

### 7.5 Proof of lemma 3.1

*Proof.* Let $T$ be the first instant such that $x_T = 0$. We know that this implies $x_t = 0, \forall t \geq T$. Since $u'(0) = +\infty$, we have $y_t > 0$, $\forall t \geq T$.
We have from the proof of proposition (3.3) that $\mu_t = 0, \forall t \geq T$. Thus, if $\lambda_t = 0$, for some $t \geq T$, we have:

$$\begin{aligned} c_r = u^{'}(y_t) &= c_e + \lambda_t - \gamma_t^x \\ &= c_e - \gamma_t^x \\ &\leq c_e < c_r, \end{aligned}$$

a contradiction. Hence $\lambda_t > 0, \forall t \geq T$.

We now prove that $\lambda_t > 0$, $\forall t \leq T$. Observe that $x_t > 0$, $\forall t \leq T - 1$. We have from equation (11) that

$$\lambda_{T-1} = \beta\lambda_T + \gamma_{T-1}^s > 0,$$

and

$$\begin{aligned}
x_{T-1} &= S_{T-2} - S_{T-1} > 0 \\
\implies S_{T-2} &> 0 \\
\implies \gamma^s_{T-2} &= 0 \\
\implies 0 < \lambda_{T-1} &= \frac{\lambda_{T-2}}{\beta} \\
\implies \lambda_{T-2} &> 0.
\end{aligned}$$

By induction we have $\lambda_t > 0, \forall t \leq T$. We end the proof.

□

### 7.6 Proof of Theorem 1

*Proof.* Let $T > 0$ be the first date when $x_T = 0$. Then, from the previous results, $x_t = a_t = 0,\ \forall t \geq T$. We have $S_t = \bar{S},\ \forall t \geq T-1,\ y_t > 0,\ \forall t \geq T$. We also have $Z_T = \frac{Z_{T-1}}{1+\alpha}$. Assume $\bar{S} > 0$. We can choose $\epsilon > 0$ which satisfies $y_T - \epsilon > 0,\ \frac{\bar{Z}+\zeta\epsilon}{1+\alpha} < \bar{Z}, \bar{S} - \epsilon > 0$. Define a sequence $(x'_t, y'_t, a'_t, S'_t, Z'_t)_t$ as follows:

$$\forall t \leq T-1,\ x'_t = x_t,\ y'_t = y_t,\ a'_t = a_t,\ S'_t = S_t\ , Z'_t = Z_t$$

$$x'_T = \epsilon,\ y'_T = y_T - \epsilon,\ a'_T = a_T = 0,\ S'_T = \bar{S} - \epsilon,\ Z'_T = Z_T + \frac{\zeta\epsilon}{1+\alpha} = \frac{Z_{T-1}+\zeta\epsilon}{1+\alpha}$$

$$\forall t \geq T+1,\ x'_t = x_t = 0,\ y'_t = y_t,\ a'_t = a_t = 0,\ S'_t = \bar{S} - \epsilon,\ Z'_t = \frac{Z'_{t-1}}{1+\alpha}$$

We claim that the sequence $(x'_t, y'_t, a'_t, S'_t, Z'_t)_t$ is feasible.

– (i) Obviously it is true for $t \leq T-1$.

– (ii) Let us check for $T$. We have

$$\begin{aligned}
S'_T &= \bar{S} - \epsilon > 0 \\
S'_{T-1} - S'_T &= \bar{S} - (\bar{S} - \epsilon) = \epsilon = x'_T \\
y'_T &= y_T - \epsilon > 0 \\
Z'_T - Z'_{T-1} - \zeta x'_T + a_T + \alpha Z'_T &= Z_T + \frac{\zeta\epsilon}{1+\alpha} - Z_{T-1} - \zeta\epsilon + \alpha(Z_T + \frac{\zeta\epsilon}{1+\alpha}) \\
&= (1+\alpha)Z_T - Z_{T-1} = 0 \\
0 < Z'_T &\leq \frac{\bar{Z}+\zeta\epsilon}{1+\alpha} < \bar{Z}
\end{aligned}$$

– (iii) We now check for $t \geq T+1$.

$$\begin{aligned}
S'_t &= \bar{S} - \epsilon > 0 \\
S'_t - S'_{t+1} &= 0 = x'_t \\
y'_t &= 0 \\
a'_t &= 0 \\
Z'_t - Z'_{t-1} - \zeta x'_t + a_t + \alpha Z'_t &= Z'_t - Z'_{t-1} + \alpha Z'_t = 0 \\
0 < Z'_t &= \frac{Z'_{t-1}}{1+\alpha} \leq \frac{Z'_T}{1+\alpha} < \frac{\bar{Z}}{1+\alpha} < \bar{Z}
\end{aligned}$$

Our claim is true. We obtain a contradiction:

$$\sum_{t=0}^{\infty}\beta^t[u(x'_t+y'_t)-c_e x'_t-c_a a'_t-c_r y'_t]-\sum_{t=0}^{\infty}\beta^t[u(x_t+y_t)-c_e x_t-c_a a_t-c_r y_t]=\beta^T(c_r-c_e)\epsilon>0$$

Therefore $\bar{S}=0$. □

## 7.7 Proof of proposition 3.4

*Proof.* Suppose $a_t>0$, then we have from equations (12) and (9) that $\gamma_t^a=0$ and $\mu_t=c_a\geq\mu_{t+1}$. We have from equation (10) that

$$\begin{aligned}\eta_t=\gamma_t^z+(1+\alpha)\mu_t-\beta\mu_{t+1} &= \gamma_t^z+(1+\alpha)c_a-\beta\mu_{t+1}\\ &= \gamma_t^z+\beta(c_a-\mu_{t+1})+(1+\alpha-\beta)c_a>0.\end{aligned}$$

Equation (13) implies that $Z_t=\bar{Z}$. Hence the proof. □

## 7.8 Proof of proposition 3.5

*Proof.* Suppose $a_t>0$. Then we have $x_t>0$, $\gamma_t^a=0$.This implies, from equation (9), that $\mu_t=c_a$.
From equations (7), (8) we obtain:

$$\begin{aligned}& -c_e-\zeta c_a-\lambda_t+c_r-\gamma_t^y=0\\ \Leftrightarrow\quad & c_r-c_e-\zeta c_a=\lambda_t+\gamma_t^y>0\\ \Rightarrow\quad & c_r>c_e-\zeta c_a.\end{aligned}$$

□

## 7.9 Proof of proposition 3.6

*Proof.* The claim is true for $t\geq T$. Let $t<T$. Then $x_t>0$.
(i) If $Z_t<\bar{Z}$ then proposition 3.4 implies $a_t=0$.
(ii) Consider the case $Z_t=\bar{Z}$. We have

$$\begin{aligned}& x_t=\frac{\bar{Z}(1+\alpha)-Z_{t-1}+a_t}{\zeta}\geq\frac{\alpha\bar{Z}}{\zeta}\\ \Rightarrow\quad & u'(x_t)\leq u'(\frac{\alpha\bar{Z}}{\zeta})\end{aligned}$$

If $a_t>0$ then $\mu_t=c_a$ (FOC (9)). (7) implies

$$u'(x_t+y_t)=c_e+\zeta c_a+\lambda_t>c_e+\zeta c_a>u'(\frac{\alpha\bar{Z}}{\zeta})$$

We get a contradiction

$$u'(\frac{\alpha\bar{Z}}{\zeta})<c_e+\zeta c_a<u'(x_t+y_t)\leq u'(x_t)\leq u'(\frac{\alpha\bar{Z}}{\zeta})$$

□

## 7.10 Proof of proposition 3.7

*Proof.* Let $T$ be the first instant when $x_T = 0$. Suppose there exists $t$ such that $Z_t = 0$. Then $a_t = 0$ (from corollary 3.1). And we have from equation (2) that

$$(1+\alpha)Z_t - Z_{t-1} = \zeta x_t \quad \text{and} \quad Z_t = 0 \text{ implies that } - Z_{t-1} = \zeta x_t = 0.$$

This is impossible for $t < T$ because $x_t > 0$. Thus $Z_t > 0, \forall t < T$.
On other hand $(1+\alpha)Z_T - Z_{T-1} = 0$ implies that $Z_T > 0$. And for $t > T$ we have $(1+\alpha)Z_t - Z_{t-1} = 0 \Rightarrow Z_t = Z_{t-1}/(1+\alpha)$. Thus rearranging these equation we find $Z_t = Z_T/(1+\alpha)^{t-T}$, that implies $Z_t > 0$. Hence the proof. □

## 7.11 Proof of proposition 3.8

*Proof.* First, observe that $S_t > 0, \gamma_t^s = 0, \ \forall t < T-1$. Suppose $Z_{T-2} < \bar{Z}$ and $y_{T-2} > 0$. In this case, we have $\gamma_{T-2}^y = 0, \ \eta_{T-2} = 0$.

$$\begin{aligned}\text{From equations (7), (8), } c_r = u^{'}(x_{T-2} + y_{T-2}) &= c_e + \zeta\mu_{T-2} + \lambda_{T-2} \\ c_r \geq u^{'}(x_{T-1} + y_{T-1}) &= c_e + \zeta\mu_{T-1} + \lambda_{T-1}.\end{aligned}$$

Therefore

$$\begin{aligned} c_e(1-\beta) < c_r(1-\beta) &\leq u^{'}(x_{T-2} + y_{T-2}) - \beta u^{'}(x_{T-1} + y_{T-1}) \\ &= (1-\beta)c_e + \zeta(\mu_{T-2} - \beta\mu_{T-1}) + \lambda_{T-2} - \beta\lambda_{T-1} \\ &\quad \text{(from equation (7))} \\ &= (1-\beta)c_e + \zeta(\frac{\beta}{1+\alpha}\mu_{T-1} + \frac{\eta_{T-2}}{1+\alpha} - \beta\mu_{T-1}) + \gamma_{T-2}^s \\ &= (1-\beta)c_e + \zeta(\frac{\beta}{1+\alpha}\mu_{T-1} - \beta\mu_{T-1}) \\ &= (1-\beta)c_e - \zeta\frac{\beta\alpha}{1+\alpha}\mu_{T-1} \leq (1-\beta)c_e, \end{aligned}$$

which is a contradiction. Hence $y_{T-2} > 0 \implies Z_{T-2} = \bar{Z}$. By induction we find $y_{T-t} > 0 \implies Z_{T-t} = \bar{Z}$ for $t > 2$ □

## 7.12 Proof of proposition 3.9

*Proof.* For $t < T$ we have $x_t > 0, S_t > 0$. Let $t < T-2$, then $x_t > 0, x_{t+1} > 0$. Suppose $y_t > 0, y_{t+1} = 0$.
$y_t > 0 \implies \gamma_t^y = 0$, thus we have from equation (8):

$$\begin{aligned} u^{'}(x_t + y_t) &= c_r \\ u^{'}(x_{t+1}) = u^{'}(x_{t+1} + y_{t+1}) &= c_r - \gamma_{t+1}^a \end{aligned}$$

that implies

$$u^{'}(x_t + y_t) \geq u^{'}(x_{t+1}) \tag{23}$$

$$\text{equivalently} \quad x_t + y_t \leq x_{t+1}$$

$$\implies \quad x_{t+1} > x_t \tag{24}$$

On other hand we have

$$
\begin{aligned}
& y_t > 0 \implies Z_t = \bar{Z}(\text{ Proposition 3.8}) \\
\implies \quad & 0 \geq Z_{t+1} - Z_t = \zeta x_{t+1} - a_{t+1} - \alpha Z_{t+1} \\
\implies \quad & x_{t+1} \leq \frac{\alpha}{\zeta} Z_{t+1} + \frac{a_{t+1}}{\zeta} \leq \frac{\alpha}{\zeta}\bar{Z} + \frac{a_{t+1}}{\zeta}
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
Z_t = \bar{Z} \qquad & \implies 0 \leq Z_t - Z_{t-1} = \zeta x_t - a_t - \alpha Z_t \\
& \implies x_t \geq \frac{\alpha}{\zeta} Z_t + \frac{a_t}{\zeta} \geq \frac{\alpha}{\zeta}\bar{Z}.
\end{aligned} \tag{26}
$$

Relations (24), (25) and (26) imply

$$
\begin{aligned}
a_{t+1} > 0 \qquad & \implies \gamma_a^{t+1} = 0 \\
& \implies \mu_{t+1} = c_a \geq \mu_t.
\end{aligned} \tag{27}
$$

We have from equation (7)

$$
\begin{aligned}
u^{'}(x_t + y_t) &= c_e + \zeta\mu_t + \lambda_t \\
u^{'}(x_{t+1}) = u^{'}(x_{t+1} + y_{t+1}) &= c_e + \zeta\mu_{t+1} + \lambda_{t+1} \\
\text{hence, from (27)} & \\
u^{'}(x_{t+1}) - u^{'}(x_t + y_t) &= \zeta(\mu_{t+1} - \mu_t) + \lambda_{t+1} - \lambda_t \\
&\geq \lambda_{t+1} - \lambda_t
\end{aligned} \tag{28}
$$

Since $S_t > 0$ we have $\gamma_t^S = 0$ and $\lambda_t = \beta\lambda_{t+1}$ (equation (11)). From Proposition (3.1), $\lambda_t > 0$ and consequently, $\lambda_{t+1} = \frac{\lambda_t}{\beta} > \lambda_t$. Relation (28) implies

$$u^{'}(x_{t+1}) > u^{'}(x_t + y_t).$$

We obtain a contradiction with relation (23). Thus the proof. □

### 7.13 Proof of proposition 3.10

*Proof.* Assume $u^{'}(\frac{\alpha}{\zeta}\bar{Z}) \leq c_r$.
Let $t < T - 1$. If $x_t > 0, y_t > 0$, then $Z_t = \bar{Z}$ (from proposition 3.8) and $u^{'}(x_t + y_t) = c_r$ (from equations (8) and (16)). Thus we have from equation (2)

$$
\begin{aligned}
& 0 \leq Z_t - Z_{t-1} = \zeta x_t - \alpha Z_t - a_t \\
\implies \quad & x_t \geq \frac{\alpha}{\zeta}\bar{Z} + a_t \\
\implies \quad & x_t + y_t \geq \frac{\alpha}{\zeta}\bar{Z} + y_t + a_t > \frac{\alpha}{\zeta}\bar{Z} \\
\implies \quad & c_r = u^{'}(x_t + y_t) \leq u^{'}(\frac{\alpha}{\zeta}\bar{Z} + y_t + a_t) < u^{'}(\frac{\alpha}{\zeta}\bar{Z}) \leq c_r,
\end{aligned}
$$

which is a contradiction. Hence we cannot have together $x_t > 0, y_t > 0$. Since $x_t > 0$ for $t < T$, we have $y_t = 0$ for $t \leq T - 2$.

□

## 7.14 Proof of proposition 3.11

*Proof.* Let $\tau$ be the first instant when $y_\tau > 0$ ($y_{\tau-1} = 0$). We know from proposition 3.9 that $y_\sigma > 0$, for any $\sigma \geq \tau$. Suppose $a_t > 0$ for for some $t$, $\tau + 1 \leq t \leq T - 2$.
Take $\epsilon > 0$ which satisfies

$$S_\sigma - \epsilon > 0, \text{ for } \sigma \in \{\tau, \tau+1, \ldots, t-1\}, \ y_\tau - \epsilon > 0, \ x_t - \epsilon > 0, \ a_t - \zeta\epsilon > 0$$

Define a sequence $(x'_\sigma, \ y'_\sigma, \ a'_\sigma, \ S'_\sigma, \ Z'_\sigma, \ )$ as follows

$$\begin{aligned}
&x'_\tau = x_\tau + \epsilon \\
&y'_\tau = y_\tau - \epsilon \\
&a'_\tau = a_\tau + \zeta\epsilon \\
&S'_\sigma = S_\sigma - \epsilon, \sigma \in \{\tau, \tau+1, \ \ldots, t-1\} \\
&x'_\sigma = x_\sigma, y'_\sigma = y_\sigma, \ a'_\sigma = a_\sigma, \ \sigma \in \{\tau+1, \ldots, t-1\}, \text{ if } t-1 \geq \tau+1 \\
&x'_t = x_t - \epsilon \\
&y'_t = y_t + \epsilon \\
&a'_t = a_t - \zeta\epsilon \\
&\ \text{for } \sigma \geq t+1, \ x'_\sigma = x_\sigma, \ y'_\sigma = y_\sigma, \ a'_\sigma = a_\sigma, \ S'_\sigma = S_\sigma, \\
&\ \text{for } \sigma \geq 0, \ Z'_\sigma(1+\alpha) - Z'_{\sigma-1} + a'_\sigma = \zeta x'_\sigma
\end{aligned}$$

One can easily check that the sequence $(x'_\sigma, \ y'_\sigma, \ a'_\sigma, \ S'_\sigma, \ Z'_\sigma, \ )$ is feasible from $Z_{-1}$. However , the difference between utilities $\triangle_\epsilon$ generated by the new and and the old sequences is strictly positive. Indeed,

$$\begin{aligned}
\triangle_\epsilon \quad &= \beta^\tau[u(x_\tau + \epsilon + y_\tau - \epsilon) - c_e(x_\tau + \epsilon) - c_a(a_\tau + \zeta\epsilon) - c_r(y_\tau - \epsilon)] \\
&+\beta^t[u(x_t - \epsilon + y_t + \epsilon) - c_e(x_t - \epsilon) - c_a(a_t - \epsilon) - c_r(y_t + \epsilon)] \\
&-\beta^\tau[u(x_\tau + y_\tau) - c_e x_\tau - c_a a_\tau - c_r y_\tau] \\
&-\beta^t[u(x_t + y_t) - c_e x_t - c_a a_t - c_r y_t] \\
&= \epsilon\beta^\tau[(c_r - c_e - \zeta c_a) - \beta^{t-\tau}(c_r - c_e - \zeta c_a)] > 0.
\end{aligned}$$

We obtain a contradiction. Hence, $a_t = 0$. □

## 7.15 Proof of proposition 3.12

*Proof.* $(i)$ Suppose $x_t > 0, Z_{t-1} = \bar{Z}, Z_t < \bar{Z}$. We have from proposition 3.4 that $a_t = 0$. And we have from equation (2) that

$$0 > Z_t - Z_{t-1} = \zeta x_t - \alpha Z_t \implies x_t < \alpha Z_t/\zeta = \alpha\bar{Z}/\zeta. \tag{29}$$

Observe that, since $x_t > 0$, we have $Z_t > 0$.
If $x_{t+1} = 0$ then $x_\tau = 0 < \alpha\bar{Z}/\zeta$ and $Z_\tau < \bar{Z}, \forall\tau \geq t$.

Now, suppose $x_{t+1} > 0$. We have

$$\begin{aligned} Z_t < \bar{Z} &\implies \eta_t = 0 \quad \text{(from equation (13))} \\ &\implies \mu_{t+1} = \frac{(1+\alpha)}{\beta}\mu_t \quad \text{from equation (10),} \\ x_{t+1} > 0 &\implies S_t > 0 \implies \gamma_t^s = 0 \\ &\implies \lambda_{t+1} = \frac{\lambda_t}{\beta} \implies \lambda_{t+1} > \lambda_t > 0 \text{ (lemma 3.1)} \end{aligned}$$

On the other hand we have:

$$\begin{aligned} u^{'}(x_t + y_t) &= c_e + \zeta\mu_t + \lambda_t \\ u^{'}(x_{t+1} + y_{t+1}) &= c_e + \zeta\mu_{t+1} + \lambda_{t+1} \\ \lambda_{t+1} > \lambda_t, \mu_{t+1} \geq \mu_t &\implies c_e + \zeta\mu_t + \lambda_t < c_e + \zeta\mu_{t+1} + \lambda_{t+1} \\ \implies u^{'}(x_t + y_t) &< u^{'}(x_{t+1} + y_{t+1}) \leq c_r \\ \implies y_t = 0 \quad &\text{and} \quad x_{t+1} \leq x_{t+1} + y_{t+1} < x_t \end{aligned} \tag{30}$$

If $Z_{t+1} = \bar{Z}$ then we will have

$$\begin{aligned} 0 < Z_{t+1} - Z_t &= \zeta x_{t+1} - a_{t+1} - \alpha Z_{t+1} \\ \implies x_{t+1} &> (a_{t+1} + \alpha\bar{Z})/\zeta \geq \alpha\bar{Z}/\zeta \end{aligned} \tag{31}$$

Relations (29), (30) and (31) give a contradiction. Hence by induction we have $x_\tau < \alpha\bar{Z}/\zeta$, $Z_\tau < \bar{Z}, \forall\tau \geq t$.

□

### 7.16 Proof of proposition 3.13

*Proof.* Suppose $x_t > 0, Z_{t-1} < \bar{Z}, Z_t = \bar{Z}$. We have

$$0 < Z_t - Z_{t-1} = \zeta x_t - a_t - \alpha Z_t \implies x_t > (a_t + \alpha Z_t)/\zeta \geq \alpha Z_t/\zeta = \alpha\bar{Z}/\zeta \tag{32}$$

Suppose $Z_{t-2} = \bar{Z}$. Then from the first part $(i)$ of this proposition we have $Z_\tau < \bar{Z}, \forall\tau \geq t-1$. A contradiction because we have $Z_t = \bar{Z}$. Hence we have $Z_\tau < \bar{Z}, \forall\tau : 0 \leq \tau \leq t-1$.

Let $0 \leq \tau < t$. We have, from just above, that $Z_\tau < \bar{Z}$, which implies $\eta_\tau = 0$. Since $x_t > 0$, we have $x_\tau > 0$ hence $Z_\tau > 0$, and $\mu_{\tau+1} = \frac{1+\alpha}{\beta}\mu_\tau$.
We have also that $0 < x_{\tau+1} = S_\tau - S_{\tau+1}, \implies S_\tau > 0, \implies \gamma_\tau^s = 0 \implies \lambda_{\tau+1} = \frac{\lambda_\tau}{\beta} \implies \lambda_{\tau+1} > \lambda_\tau$ ($\lambda_\tau > 0$ from proposition 3.1).
We have from equation (7):

$$\begin{aligned} u^{'}(x_\tau + y_\tau) &= c_e + \zeta\mu_\tau + \lambda_\tau \\ &< c_e + \zeta\mu_{\tau+1} + \lambda_{\tau+1} = u^{'}(x_{\tau+1} + y_{\tau+1}) \leq c_r \\ &\implies y_\tau = 0 \\ &\implies x_\tau > x_{\tau+1} + y_{\tau+1} \geq x_{\tau+1} \end{aligned}$$

By induction, $x_\tau > x_{\tau+1}$, for $0 \leq \tau < t$.

□

## 7.17 Proof of proposition 3.14

*Proof.* Suppose $Z_t = \bar{Z}$ and $Z_{t+1} < \bar{Z}$.
Since $Z_{t+1} < \bar{Z}$ we have $y_{t+1} = 0$, hence $x_{t+1} > 0$.
We have from proposition 3.4 that $a_{t+1} = 0$, and

$$\begin{aligned} 0 > Z_{t+1} - \bar{Z} &= Z_{t+1} - Z_t \\ &= \zeta x_{t+1} - \alpha Z_{t+1} \\ \implies x_{t+1} &< \alpha Z_{t+1}/\zeta < \alpha\bar{Z}/\zeta \\ \implies c_r \geq u^{'}(x_{t+1} + y_{t+1}) &= u^{'}(x_{t+1}) > u^{'}(\alpha\bar{Z}/\zeta) \geq c_r, \end{aligned}$$

a contradiction. By induction, if $Z_t = \bar{Z}$, then $Z_\tau = \bar{Z}$ for $\tau \geq t$ such that $x_\tau > 0$. Hence the proof. □

## 7.18 Proof of proposition 3.15

*Proof.* Suppose we have
$Z_t = Z_{t+1} = \bar{Z}$ and $a_{t+1} > 0$. Since $t \leq T-2$, we have $x_t > 0, x_{t+1} > 0$.
We have $a_{t+1} > 0 \implies \gamma^a_{t+1} > 0 \implies \mu_{t+1} = c_a \geq \mu_t$, and $a_{t+1} > 0 \implies y_t = 0$ (from proposition 3.11).
We have also that

$$\begin{aligned} x_{t+1} = S_t - S_{t+1} > 0 \implies S_t > 0 \implies \gamma^s_t > 0 \\ \implies \lambda_{t+1} = \frac{\lambda_t}{\beta} \geq \lambda_t \implies \lambda_{t+1} > \lambda_t \end{aligned}$$

that gives

$$\begin{aligned} u^{'}(x_t) \quad &= \quad c_e + \zeta\mu_t + \lambda_t \\ &< \quad c_e + \zeta\mu_{t+1} + \lambda_{t+1} = u^{'}(x_{t+1} + y_{t+1}) \\ \text{thus} \quad & \quad u^{'}(x_t) < u^{'}(x_{t+1} + y_{t+1}) \\ &\iff \quad x_t > x_{t+1} + y_{t+1} \geq x_{t+1} \end{aligned} \tag{33}$$

If $Z_{t-1} = \bar{Z}$ then

$$\begin{aligned} 0 = Z_t - Z_{t-1} \quad &= \quad \zeta x_t - \alpha Z_t - a_t \quad \text{(from equation (2))} \\ &\implies \quad x_t = \alpha\bar{Z}/\zeta + a_t/\zeta \end{aligned} \tag{34}$$

But we have also that

$$\begin{aligned} Z_{t+1} - Z_t \quad &= \quad \zeta x_{t+1} - a_{t+1} - \alpha Z_{t+1} = 0 \\ &\implies \quad x_{t+1} = \alpha\bar{Z}/\zeta + a_{t+1}/\zeta \end{aligned} \tag{35}$$

Equations (33), (34) and (35) imply

$$a_t > a_{t+1} > 0. \tag{36}$$

Take $\epsilon > 0$ sufficiently low. Define a sequence $(x'_s, a'_s)_s$ by

$$\begin{aligned}
&x'_s = x_s,\ a'_s = a_s,\ \forall s \neq t \\
&x_t^{'} = x_t - \epsilon \\
&a_t^{'} = a_t - \epsilon\zeta \\
&x_{t+1}^{'} = x_{t+1} - \epsilon \\
&a_{t+1}^{'} = a_{t+1} - \epsilon\zeta
\end{aligned}$$

One can easily check that the sequences $(x_s^{'})$ and $(a_s^{'})$ are feasible from $Z_{-1}, S_{-1}$.

However, for $\epsilon$ small enough, the difference between utilities $\triangle_\epsilon$ generated by the new and the old sequences is strictly positive. Indeed,

$$\begin{aligned}
\triangle_\epsilon \quad & = \beta^t[u(x_t - \epsilon) - c_e(x_t - \epsilon) - c_a(a_t - \zeta\epsilon)] \\
& +\beta^{t+1}[u(x_{t+1} + \epsilon) - c_e(x_{t+1} + \epsilon) - c_a(a_{t+1} + \epsilon)] \\
& -\beta^t[u(x_t) - c_e x_t - c_a a_t] \\
& -\beta^{t+1}[u(x_{t+1}) - c_e x_{t+1} - c_a a_{t+1}] \\
& = \epsilon\beta^t[(u(x_t - \epsilon) - u(x_t)) + (u(x_{t+1} + \epsilon) - u(x_{t+1})) \\
& +(c_e + \zeta c_a) - \beta(c_e + \zeta c_a)] \\
& > \epsilon\beta^t[(u(x_t - \epsilon) - u(x_t)) + (u(x_{t+1} + \epsilon) - u(x_{t+1})) \\
& \geq \epsilon\beta^t[u^{'}(x_{t+1}) - u^{'}(x_t)] > 0, \text{ when } \epsilon > 0 \text{ is small enough}
\end{aligned}$$

Thus we obtain a contradiction. Hence $Z_{t-1} < \bar{Z}$.

□

## 7.19 Proof of theorem 2

*Proof.* We consider case (ii). From the very definition of $\tau$, $Z_t < \bar{Z}$ for any $t < \tau - 1$.
$\tau'$ is the greatest date less than $T$ for which $Z_t = \bar{Z}$. From Proposition 3.14, if $c_r \leq u'(\frac{\alpha\bar{Z}}{\zeta})$, then $\tau' = T - 1$. From Proposition 3.12, $Z_t < \bar{Z}, \forall t \geq \tau'$. □

## 7.20 Proof of proposition 4.1

*Proof.* Define $\underline{S} = \frac{\alpha\bar{Z}}{\zeta}$. Take $S_{-1} < \underline{S}$. We have

$$\begin{aligned}
& Z_{T-1}(1+\alpha) - Z_{T-2} = \zeta x_{T-1} - a_{T-1} = \zeta S_{T-2} - a_{T-1} < \zeta S_{-1} - a_{T-1} < \zeta \underline{S} \\
\Rightarrow \quad & Z_{T-1}(1+\alpha) < \alpha\bar{Z} + Z_{T-2} \leq (1+\alpha)\bar{Z} \\
\Rightarrow \quad & Z_{T-1} < \bar{Z}
\end{aligned}$$

But also

$$\begin{aligned}
& Z_{T-2}(1+\alpha) - Z_{T-3} = \zeta x_{T-2} - a_{T-2} = \zeta(S_{T-3} - S_{T-2}) - a_{T-1} < \zeta S_{-1} - a_{T-1} < \zeta \underline{S} \\
\Rightarrow \quad & Z_{T-2}(1+\alpha) < \alpha\bar{Z} + Z_{T-3} \leq (1+\alpha)\bar{Z} \\
\Rightarrow \quad & Z_{T-2} < \bar{Z}
\end{aligned}$$

By induction, $Z_t < \bar{Z}$, for $t < T$. Since $Z_{t+T} = \frac{Z_{T-1}}{(1+\alpha)^{1+t}}$, we have $Z_t < \bar{Z}, \forall t$. □

### 7.21 Proof of proposition 4.2

*Proof.* Consider an optimal path. Assume for any $t < T$, $Z_t < \bar{Z}$. recall that this implies $a_t = 0, \forall t$ and $y_t = 0, \forall t \leq T_2$. We have

$$\begin{aligned} Z_0(1+\alpha) - Z_{-1} &= \zeta(S_{-1} - S_0) \\ Z_1(1+\alpha) - Z_0 &= \zeta(S_0 - S_{-1}) \\ &\dots \\ Z_{T-1}(1+\alpha) - Z_{T-2} &= \zeta(S_{T-2} - S_{T-1}) = \zeta S_{T-2} \end{aligned}$$

Summing these equations we obtain:

$$\alpha(Z_0 + Z_1 + \ldots + Z_{T-2}) + Z_{T-1}(1+\alpha) - Z_{-1} = \zeta S_{-1} \tag{37}$$

This implies

$$S_{-1} \leq \frac{\bar{Z}(\alpha T + 1)}{\zeta} \tag{38}$$

On the other hand, FOC (8) implies

$$u'(x_t + y_t) \leq u'(x_t) \leq c_r$$

Hence

$$u'\left(\frac{Z_t(1+\alpha) - Z_{t-1}}{\zeta}\right) \leq c_r$$

equivalently

$$Z_t(1+\alpha) - Z_{t-1} \geq \zeta {u'}^{-1}(c_r)$$

More explicitly

$$\begin{aligned} Z_0(1+\alpha) - Z_{-1} &\geq \zeta {u'}^{-1}(c_r) \\ Z_1(1+\alpha) - Z_0 &\geq \zeta {u'}^{-1}(c_r) \\ &\dots \\ Z_{T-2}(1+\alpha) - Z_{T-3} &\geq \zeta {u'}^{-1}(c_r) \end{aligned}$$

Sum these inequalities.

$$\begin{aligned} (T-1)\zeta {u'}^{-1}(c_r) &\leq \alpha(Z_0 + Z_1 + \ldots + Z_{T-3}) + Z_{T-2}(1+\alpha) - Z_{-1} \\ (T-1)\zeta {u'}^{-1}(c_r) &\leq \alpha(Z_0 + Z_1 + \ldots + Z_{T-3} + Z_{T-2}) + Z_{T-2} - Z_{-1} \\ (T-1)\zeta {u'}^{-1}(c_r) &\leq \alpha(Z_0 + \ldots + Z_{T-2}) + Z_{T-1}(1+\alpha) - Z_{-1} \end{aligned} \tag{39}$$

Relations (37), (38), (39) imply

$$(T-1)\zeta {u'}^{-1}(c_r) \leq \zeta S_{-1} \leq \bar{Z}(\alpha T + 1) \tag{40}$$

One can check that under the assumption $c_r < u'(\frac{\alpha \bar{Z}}{\zeta})$, that

$$T \leq \bar{T} = \frac{{u'}^{-1}(c_r) + \bar{Z}/\zeta}{{u'}^{-1}(c_r) - (\bar{Z}\alpha)/\zeta}$$

$$S_{-1} \leq \bar{S} = \frac{\bar{Z}}{\zeta}(\alpha \bar{T} + 1)$$

Hence, if $S_{-1} > \bar{S}$, any optimal path starting from $S_{-1}$ will have $Z_t = \bar{Z}$ at some $t$. □

### 7.22 Proof of Theorem 3

*Proof.* Let

$$S_s = \sup\{\underline{S} : S_{-1} < \underline{S} \Rightarrow \text{any optimal path from } S_{-1} \text{ satisfies } Z_t < \bar{Z}, \forall t\}$$

and

$$S^s = \inf\{\bar{S} : S_{-1} < \underline{S} \Rightarrow \text{any optimal path from } S_{-1} \text{ satisfies } Z_t = \bar{Z}, \text{ for some } t\}$$

It is easy to prove $S^s = S_s$. The critical value is $S^h_{-1} = S^s = S_s$. □

### 7.23 Proof of Theorem 4

*Proof.* Before proving (a),...,(e), we will prove that $\mu_t = 0, \forall t$. Indeed, since $Z_t < \bar{Z}, \forall t$, we have $\eta_t = 0$, $\forall t$. Also, from proposition 3.7, we have $Z_t > 0, \forall t$, hence $\gamma^z_t = 0, \forall t$. Using FOC (10), we get $\mu_{t+1} = (1+\alpha)\mu_t/\beta, \forall t$. If $\mu_\tau > 0$ for some $\tau$, then $\mu_t \to +\infty$ when $t \to +\infty$. That contradicts FOC (9) imposing $\mu_t \leq c_a, \forall t$. Hence $\mu_t = 0, \forall t$.
We now pass to the proof of (a),( b), (c), (d), (e).
(a) It follows Proposition 3.4.
(b) It follows Theorem 1.
(c) It follows the fact that $x_t = 0, a_t = 0$ for $t \geq T$.
(d) 1. First, we prove $y_{T-t} = 0 \implies y_{T-t-1} = 0$ for $t \geq 1$.
Suppose the contrary $y_{T-t} = 0, y_{T-t-1} > 0$. We have

$$\begin{aligned} u'(x_{T-t} + y_{T-t}) &= c_r - \gamma^y_{T-t} \\ &= c_e + \lambda_{T-t} \text{ since } x_{T-t} > 0,\ \mu_{T-t} = 0. \end{aligned}$$

and

$$\begin{aligned} u'(x_{T-t-1} + y_{T-t-1}) &= c_r \\ &= c_e + \lambda_{T-t-1} \\ &= c_e + \beta\lambda_{T-t} \text{ since } S_{T-t-1} > 0 \text{ and FOC (11)} \end{aligned}$$

That implies, since $\lambda_t > 0$ (lemma 3.1)

$$c_r = c_e + \beta\lambda_{T-t} < c_e + \lambda_{T-t} \leq c_r$$

a contradiction.
2. We now prove that we cannot have both $y_{T-t} > 0,\ y_{T-t-1} > 0$. Suppose the contrary. Then

$$\begin{aligned} u'(x_{T-t} + y_{T-t}) &= c_r \\ &= c_e + \lambda_{T-t} \\ u'(x_{T-t-1} + y_{T-t-1}) &= c_r \\ &= c_e + \lambda_{T-t-1} \\ &= c_e + \beta\lambda_{T-t} \end{aligned}$$

That yields a contradiction. Hence, either $y_{T-t} > 0$, $y_{T-t-1} = 0$, or $y_{T-t} = 0$, $y_{T-t-1} > 0$. The second case is impossible from point 1. Summing up
(i) either $y_{T-1} > 0$ and $y_{T-2} = 0$ and by point 1, $y_{T-t} = 0, \forall t \leq T-2$
(ii) or $y_{T-1} = 0$ and point 1 implies $y_t = 0, \forall t \leq T-2$.
For $t \geq T$, since $x_t = 0$, we have $y_t > 0$. Use FOC (8).
(e) It follows the fact that $a_t = 0$ for $t \leq T$.
(f) Consider $t \in \{0, 1, \ , T-2\}$. FOC (11) implies $\lambda_t = \beta\lambda_{t+1}$, while FOC (7) implies (with $\mu_\tau = 0, \ \forall \tau$)

$$\begin{aligned} u'(x_t) = u'(x_t + y_t) &= c_e + \lambda_t \\ u'(x_{t+1} + y_{t+1}) &= c_e + \lambda_{t+1} \end{aligned}$$

Hence

$$c_e = u'(x_t) - \lambda_t = u'(x_{t+1}) - \lambda_{t+1}$$

or equivalently

$$u'(x_t) = u'(x_{t+1}) + (\lambda_t - \lambda_{t+1}) = u'(x_{t+1}) - (1-\beta)\lambda_{t+1} < u'(x_{t+1})$$

Hence

$$x_{t+1} < x_t$$

Finally

$$x_0 > x_1 > \ldots, x_{T-1} > x_T = 0$$

Since, for $t \in \{0, 1, \ , T-2\}$

$$\begin{aligned} u'(x_t) = u'(x_t + y_t) &= c_e + \lambda_t \\ u'(x_{t+1}) = u'(x_{t+1} + y_{t+1}) &= c_e + \lambda_{t+1} > u'(x_t) \end{aligned}$$

we have

$$u'(x_0) < u'(x_1) < \ldots < u'(x_{T-2}) < u'(x_{T-1} + y_{T-1}) \leq c_r = u'(y_s), \ \forall s \geq T.$$

□

## 7.24 Proof of Theorem 5

*Proof.* (a) See Theorem 2 (ii).
(b) It is a consequence of Proposition 3.8
(c) For $t < \tau$, we have

$$\begin{aligned} (1+\alpha)\mu_t &= \beta\mu_{t+1} \text{ (FOC (10))} \\ \lambda_t &= \beta\lambda_{t+1} \text{ (FOC (11))} \\ u'(x_t) = u'(x_t + y_t) &= c_e + \zeta\mu_t + \lambda_t \text{ (FOC (7))} \\ u'(x_{t+1} + y_{t+1}) &= c_e + \zeta\mu_{t+1} + \lambda_{t+1} > c_e + \zeta\mu_t + \lambda_t = u'(x_t + y_t) \end{aligned}$$

Thus

$$x_0 > x_1 > \ldots > x_{\tau-1} > x_\tau + y_\tau$$

**Part I**

(a') See Theorem 2 (ii).

(d) We first prove that $y_{\tau+1} > 0$. Observe that $x_{\tau+1} = \frac{\alpha\bar{Z}}{\zeta}$ since, from (a), $Z_\tau = Z_{\tau+1} = \bar{Z}$. Suppose $y_{\tau+1} = 0$.

Let $(x'_t,\ y'_t,\ a'_t,\ S'_t,\ Z'_t)$ be a sequence which satisfies

$$x'_t = x_t, a'_t = a_t,\ S'_t = S_t,\ Z'_t = Z_t,\ \forall t$$

and

$$y'_t = y_t,\ \forall t \neq \tau+1,\ y'_{\tau+1} = \epsilon > 0$$

Obviously this sequence is feasible from $(S_{-1},\ Z_{-1})$.

Let $\triangle_\epsilon$ denote the difference of utilities generated by $(x'_t,\ y'_t,\ a'_t,\ S'_t,\ Z'_t)$ and $(x_t,\ y_t,\ a_t,\ S_t,\ Z_t)$. One gets:

$$\triangle_\epsilon = \beta^{\tau+1}\left[u(x_{\tau+1}+\epsilon) - u(x_{\tau+1}) - c_r\epsilon\right]$$

and

$$\lim_{\epsilon\to 0} \triangle_\epsilon/\epsilon = \beta^{\tau+1}\left[u'(x_{\tau+1}) - c_r\right] = \beta^{\tau+1}\left[u'(\frac{\alpha\bar{Z}}{\zeta}) - c_r\right] > 0$$

Hence, $\triangle_\epsilon > 0$ for $\epsilon$ small enough. That is a contradiction. Thus, $y_{\tau+1} > 0$.

Proposition 3.9 implies $y_t > 0$ for $t \in \{\tau+1, \ldots, T-1\}$.

(e) From (c) we obtain

$$u'(x_0) < u'(x_1) < \ldots < u'(x_{\tau-1}) < u'(x_\tau + y_\tau) \leq c_r$$

and from (d)

$$c_r = u'(x_{\tau+1}+y_{\tau+1}) = u'(x_{\tau+2}+y_{\tau+2}) = \ldots = u'(x_{T-1}+y_{T-1}) = u'(y_s),\ \forall s \geq T$$

(f) It is clear that $a_t = 0$ for $t \geq T$. For $t < T$, we have $x_t > 0$. Use Proposition 3.5.

(f') Recall that $Z_t = \bar{Z}, \forall t \in \{\tau, \ldots, T-1\}$. If $a_t > 0$, for some $t \in \{\tau+2, \ldots, T-1\}$, then from Proposition 3.15, $Z_{t-2} < \bar{Z}$, which is impossible.

**Part II**

(d') It is a consequence of Proposition 3.10.

(e') For $t = 0, \ldots, \tau-1$, we have $Z_t < \bar{Z}$, hence $\eta_t = 0$ and $\mu_t < \mu_{t+1}$ from FOC (10). Therefore, for $t = 0, \ldots, \tau-1$,

$$u'(x_t) = c_e + \zeta\mu_t + \lambda_t < c_e + \zeta\mu_{t+1} + \lambda_{t+1} = u'(x_{t+1})$$

We have

$$x_\tau = \frac{\bar{Z}(1+\alpha) - Z_{\tau-1}}{\zeta} > \frac{\bar{Z}(1+\alpha) - \bar{Z}}{\zeta} = x_{\tau+1}$$

which implies

$$u'(x_\tau) < u'(x_{\tau+1})$$

For $t = \tau+1, \ldots, \tau'$, we have $u'(x_t) = u'(\frac{\alpha\bar{Z}}{\zeta})$.

Also,

$$x_{\tau'+1} = \frac{Z_{\tau'+1}(1+\alpha) - \bar{Z}}{\zeta} < \frac{\alpha\bar{Z}}{\zeta} = x_{\tau'}$$

hence

$$u'(x_{\tau'}) < u'(x_{\tau'+1})$$

For $t = \tau' + 1, \ldots, T-2$, $Z_t < \bar{Z}$. $S_t > 0$ and this implies $u'(x_t) < u'(x_{t+1}) \leq c_r = u'(y_s), \forall s \geq T$.
(f) Use Proposition 3.5
(e") It is clear, since $Z_t < \bar{Z}$ for $t = 0, \ldots, \tau - 1$, $t = \tau' + 1, \ldots, T-1$ and $S_t > 0$ for $t = 0, \ldots, T-2$, that

$$u'(x_0) < \ldots < u'(x_\tau),$$

and

$$u'(x_{\tau'}) < \ldots < u'(x_{T-2}) < u'(x_{T-1} + y_{T-1}) \leq c_r$$

(i) Consider the case $\tau + 2 > \tau'$.
Assume $a_{\tau+1} > 0$. Then $\mu_{\tau+1} = c_a$ and

$$u'(x_{\tau+1}) = c_e + \zeta c_a + \lambda_{\tau+1} \geq c_e + \zeta\mu_\tau + \lambda_{\tau+1} > c_e + \zeta\mu_\tau + \lambda_\tau = u'(x_\tau)$$

Assume $a_{\tau+1} = 0$. Then $x_{\tau+1} = \frac{\alpha\bar{Z}}{\zeta} < \frac{\bar{Z}(1+\alpha) - Z_{\tau-1} + a_\tau}{\zeta} = x_\tau$ and hence $u'(x_\tau) < u'(x_{\tau+1})$.
Assume $a_{\tau+1} > 0, a_\tau > 0$. Then $c_a = \mu_t = \mu_{t+1}$. And

$$u'(x_{\tau+1}) = c_e + \zeta c_a + \lambda_{\tau+1} > c_e + \zeta c_a + \lambda_\tau = u'(x_\tau)$$

Assume $a_{\tau+1} = a_\tau = 0$. Then

$$x_{\tau+1} = \frac{\alpha\bar{Z}}{\zeta} < \frac{\bar{Z}(1+\alpha) - Z_{\tau-1}}{\zeta} = x_\tau$$

and $u'(x_\tau) < u'(x_{\tau+1})$.
(ii) Consider the case $\tau' = \tau + 2$.
We have, from proposition 3.15, $a_{\tau+2} = 0$. Therefore

$$x_{\tau+2} = \frac{\alpha\bar{Z}}{\zeta} \leq \frac{\bar{Z}(1+\alpha) - \bar{Z} + a_{\tau+1}}{\zeta} = x_{\tau+1}$$

and hence

$$u'(x_{\tau+1}) \leq u'(x_{\tau+2}) = u'(x_{\tau'})$$

If $a_{\tau+1} > 0$, then $\mu_{\tau+1} = c_a$ and

$$u'(x_{\tau+1}) = c_e + \zeta c_a + \lambda_{\tau+1} \geq c_e + \zeta\mu_\tau + \lambda_{\tau+1} > c_e + \zeta\mu_\tau + \lambda_\tau = u'(x_\tau)$$

If $a_{\tau+1} = 0$, then

$$x_{\tau+1} = \frac{\alpha\bar{Z}}{\zeta} < \frac{\bar{Z}(1+\alpha) - Z_{\tau-1} + a_\tau}{\zeta} = x_\tau$$

and

$$u'(x_{\tau+1}) > u'(x_\tau)$$

Summing up

$$u'(x_\tau) < u'(x_{\tau+1}) \leq u'(x_{\tau+2}) = u'(x_{\tau'})$$

(iii) Consider the case $\tau + 2 < \tau'$.
Then, from proposition 3.15,

$$a_{\tau'} = a_{\tau'-1} = \ldots = a_{\tau+3} = a_{\tau+2} = 0$$

This implies

$$x_{\tau'} = x_{\tau'-1} = \ldots = x_{\tau+3} = x_{\tau+2} = \frac{\alpha \bar{Z}}{\zeta}$$

equivalently

$$u'(x_{\tau'}) = u'(x_{\tau'-1}) = \ldots = u'(x_{\tau+3}) = u'(x_{\tau+2}) = u'(\frac{\alpha \bar{Z}}{\zeta})$$

Now

$$x_{\tau+2} = \frac{\alpha \bar{Z}}{\zeta} \leq \frac{\bar{Z}(1+\alpha) - \bar{Z} + a_{\tau+1}}{\zeta} = x_{\tau+1} \Rightarrow u'(x_{\tau+1}) \leq u'(x_{\tau+2})$$

Now compare $u'(x_{\tau+1})$ and $u'(x_\tau)$. If $a_{\tau+1} > 0$, then $\mu_{\tau+1} = c_a$ and

$$u'(x_{\tau+1}) = c_e + \zeta c_a + \lambda_{\tau+1} \geq c_e + \zeta \mu_\tau + \lambda_{\tau+1} > c_e + \zeta \mu_\tau + \lambda_\tau = u'(x_\tau)$$

If $a_{\tau+1} = 0$, then

$$x_{\tau+1} = \frac{\alpha \bar{Z}}{\zeta} < \frac{\bar{Z}(1+\alpha) - Z_{\tau-1} + a_\tau}{\zeta} = x_\tau$$

and

$$u'(x_{\tau+1}) > u'(x_\tau)$$

Summing up

$$u'(x_\tau) < u'(x_{\tau+1}) \leq u'(x_{\tau+2}) = u'(x_{\tau+3}) = \ldots = u'(x'_\tau) = u'(\frac{\alpha \bar{Z}}{\zeta})$$

□

### 7.25 Proof of Theorem 6

*Proof.* (i) We have, for $t \leq T-1$, $u'(x_t) \geq u'(x_t + y_t) = c_e + \zeta \mu_t + \lambda_t > c_e$. Hence $x_t < u'^{-1}(c_e), \forall t \leq T-1$. Summing from $t=0$ to $t=T-1$, we get $S_{-1} \leq T u'^{-1}(c_e)$. Hence $S_{-1} \to +\infty \Rightarrow T \to +\infty$.
(ii) From (i) we have $S_{-1} \leq T u'^{-1}(c_e)$. When $c_e \to +\infty$, from Inada Condition, $u'^{-1}(c_e) \to 0$. This implies $T \to +\infty$.
(iii) Now suppose $c_r - c_e \to +\infty$. In this case $c_r \to +\infty$. We can suppose $c_r > u'(\frac{\alpha \bar{Z}}{\zeta})$. Suppose that $T$ is bounded above. Let $T^M$ be the smallest integer which bounds $T$ when $c_r - c_e$ goes to infinity. There exists an integer $T \leq T^M$ which is the exhaustion date for an infinite sequence of optimal paths $\{(x_t^n, y_t^n, a_t^n, S_t^n, Z_t^n)_t\}_n$ associated with an infinite sequence $\{(c_r^n, c_e^n)\}_n$ which

satisfies $c_r^n - c_e^n \to +\infty$ when $n \to +\infty$. Observe, for any $n$, $\sum_{t=0}^{T-1} x_t^n = S_{-1}$. We have, for any $n$,

$$c_r^n = u'(y_T^n) \leq c_e + \zeta\mu_T^n + \lambda_T^n = c_e + \lambda_T^n$$

since $\mu_T^n = 0$. As $c_r^n$ converges to infinity, $\lambda_T^n$ converges to infinity as well. Since

$$\lambda_{T-1}^n \geq \beta\lambda_T^n,\ \lambda_{T-2}^n \geq \beta^2\lambda_T^n,\ \ldots, \lambda_0 \geq \beta^T\lambda_T^n$$

we have

$$\forall t \leq T,\ \lambda_t^n \to +\infty \text{ as } n \to +\infty$$

Since

$$u'(x_t^n + y_t^n) = c_e + \zeta\mu_t^n + \lambda_t^n$$

we have $\forall t \leq T-1,\ x_t^n + y_t^n \to 0$, and hence $x_t^n \to 0$ when $n \to +\infty$. We have a contradiction

$$S_{-1} = \sum_{t=0}^{T-1} x_t^n \to 0$$

when $n \to +\infty$. Hence $T$ must go to infinity when $c_r - c_e$ goes to infinity. □

## 7.26 Proof of Theorem 7

*Proof.* If $Z_t < \bar{Z}$ for any $t$, then $a_t = 0, \mu_t = 0, \forall t$, $y_t = 0, \forall t \leq T-2$. FOC (11) implies $\lambda_t = \beta\lambda_{t+1}, \forall t = 0, \ldots, T-2$. We have

$$\begin{aligned}
&u'(x_0) = c_e + \lambda_0 \\
&u'(x_1) = c_e + \lambda_1 = c_e + \lambda_0/\beta \\
&\ldots \\
&u'(x_{T-2}) = c_e + \lambda_{T-2} = c_e + \lambda_0/\beta^{T-2} \\
&u'(x_{T-1} + y_{T-1}) = c_e + \lambda_{T-1} = c_e + \lambda_0/\beta^{T-1}
\end{aligned}$$

Or equivalently

$$\begin{aligned}
&x_0 = {u'}^{-1}(c_e + \lambda_0) \\
&x_1 = {u'}^{-1}(c_e + \lambda_0/\beta) \\
&\ldots \\
&x_{T-2} = {u'}^{-1}(c_e + \lambda_0/\beta^{T-2}) \\
&x_{T-1} + y_{T-1} = {u'}^{-1}(c_e + \lambda_0/\beta^{T-1})
\end{aligned}$$

Summing these equalities:

$$S_{-1} + y_{T-1} = \sum_{t=0}^{T-1} {u'}^{-1}(c_e + \lambda_0/\beta^t)$$

Similarly

$$S'_{-1} + y'_{T'-1} = \sum_{t=0}^{T'-1} {u'}^{-1}(c_e + \lambda'_0/\beta^t)$$

Assume $T' \leq T-1$. We prove that $\lambda'_0 \geq \lambda_0$ in this case. Indeed, we have

$$c_r \geq u'(x_{T-1} + y_{T-1}) = c_e + \lambda_{T-1} = c_e + \lambda_0/\beta^{T-1}$$

This implies $\lambda_0 \leq \beta^{T-1}(c_r - c_e)$. But also

$$c_r = u'(y_T) \leq c_e + \lambda_T \leq c_e + \lambda_{T-1}/\beta = c_e + \lambda_0/\beta^T$$

This implies $\lambda_0 \geq \beta^T(c_r - c_e)$.
Similarly

$$\lambda'_0 \geq \beta^{T'}(c_r - c_e) \geq (c_r - c_e)\beta^{T-1} \geq \lambda_0$$

Our claim is true.

Now assume $S'_{-1} > S_{-1}$ and $T' \leq T-1$. We have

$$\begin{aligned}
S'_{-1} \leq S'_{-1} + y'_{T'-1} &= \sum_{t=0}^{T'-1} {u'}^{-1}(c_e + \lambda'_0/\beta^t) \\
&\leq \sum_{t=0}^{T'-1} {u'}^{-1}(c_e + \lambda_0/\beta^t) \leq \sum_{t=0}^{T-2} {u'}^{-1}(c_e + \lambda_0/\beta^t) \\
&= \sum_{t=0}^{T-1} {u'}^{-1}(c_e + \lambda_0/\beta^t) - u'(c_e + \lambda_0/\beta^{T-1}) \\
&= \sum_{t=0}^{T-1} {u'}^{-1}(c_e + \lambda_0/\beta^t) - u'(c_e + \lambda_{T-1}) \\
&= (S_{-1} + y_{T-1}) - (x_{T-1} + y_{T-1}) \\
&= S_{-1} - x_{T-1} < S_{-1}
\end{aligned}$$

That is a contradiction. Hence $T' > T-1$ i.e. $T' \geq T$. □

### 7.27 Proof of Theorem 8

*Proof.* If $Z_t < \bar{Z}$ for any $t$, then $a_t = 0, \mu_t = 0, \forall t$, $y_t = 0, \forall t \leq T-2$. FOC (11) implies $\lambda_t = \beta\lambda_{t+1}, \forall t = 0, \ldots, T-2$. We have

$$\begin{aligned}
&u'(x_0) = c_e + \lambda_0 \\
&u'(x_1) = c_e + \lambda_1 = c_e + \lambda_0/\beta \\
&\ldots \\
&u'(x_{T-2}) = c_e + \lambda_{T-2} = c_e + \lambda_0/\beta^{T-2} \\
&u'(x_{T-1} + y_{T-1}) = c_e + \lambda_{T-1} = c_e + \lambda_0/\beta^{T-1}
\end{aligned}$$

Or equivalently

$$\begin{aligned}
&x_0 = {u'}^{-1}(c_e + \lambda_0) \\
&x_1 = {u'}^{-1}(c_e + \lambda_0/\beta) \\
&\ldots \\
&x_{T-2} = {u'}^{-1}(c_e + \lambda_0/\beta^{T-2}) \\
&x_{T-1} + y_{T-1} = {u'}^{-1}(c_e + \lambda_0/\beta^{T-1})
\end{aligned}$$

Summing these equalities:

$$S_{-1} + y_{T-1} = \sum_{t=0}^{T-1} u'^{-1}(c_e + \lambda_0/\beta^t)$$

Moreover we have

$$c_r \geq u'(x_{T-1} + y_{T-1}) = c_e + \lambda_{T-1} = c_e + \lambda_0/\beta^{T-1}$$

This implies $\lambda_0 \leq \beta^{T-1}(c_r - c_e)$. But also

$$c_r = u'(y_T) \leq c_e + \lambda_T \leq c_e + \lambda_{T-1}/\beta = c_e + \lambda_0/\beta^T$$

This implies $\lambda_0 \geq \beta^T(c_r - c_e)$.
Let $(x'_t, y'_t, a'_t, Z'_t, S'_t)_t$ be the optimal sequence associated with $(c'_r, c'_e)$ which satisfy $c'_r - c'_e > c_r - c_e$. Let $T'$ be the exhaustion time associated with $(c'_r, c'_e)$. We then have similarly

$$S_{-1} + y'_{T'-1} = \sum_{t=0}^{T'-1} u'^{-1}(c_e + \lambda'_0/\beta^t)$$

Assume $T' \leq T-1$. We prove that $\lambda'_0 > \lambda_0$ in this case. Indeed, $\lambda'_0 \leq \beta^{T'-1}(c'_r - c'_e)$ and $\lambda'_0 \geq \beta^{T'}(c'_r - c'_e)$.
Hence

$$\lambda'_0 \geq \beta^{T'}(c'_r - c'_e) \geq (c'_r - c'_e)\beta^{T-1} > (c_r - c_e)\beta^{T-1} \geq \lambda_0$$

Our claim is true.
We have

$$\begin{aligned}
S_{-1} \leq S_{-1} + y'_{T'-1} &= \sum_{t=0}^{T'-1} u'^{-1}(c_e + \lambda'_0/\beta^t) \\
&< \sum_{t=0}^{T'-1} u'^{-1}(c_e + \lambda_0/\beta^t) \leq \sum_{t=0}^{T-2} u'^{-1}(c_e + \lambda_0/\beta^t) \\
&= \sum_{t=0}^{T-1} u'^{-1}(c_e + \lambda_0/\beta^t) - u'(c_e + \lambda_0/\beta^{T-1}) \\
&= \sum_{t=0}^{T-1} u'^{-1}(c_e + \lambda_0/\beta^t) - u'(c_e + \lambda_{T-1}) \\
&= (S_{-1} + y_{T-1}) - (x_{T-1} + y_{T-1}) \\
&= S_{-1} - x_{T-1} < S_{-1}
\end{aligned}$$

That is a contradiction. Hence $T' > T-1$ i.e. $T' \geq T$. $\square$